

Two-Sided Allocation Problems, Matching with Transfers, and the Impossibility of Ex Post Efficiency*

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Abstract

The impossibility of ex post efficient trade has been established for two-sided allocation problems with unit traders and one-dimensional types, but little is known about how this result relates to primitives or extends to more general environments. We show that the impossibility result generalizes to all two-sided allocation problems that satisfy a generalization of the complementarity condition identified by Shapley (1962). Calling a two-sided allocation problem a matching problem if it can be represented as an assignment game, we show that the impossibility result holds for all matching problems and that a necessary and sufficient condition for a two-sided allocation problem to be a matching problem is that each agent can be decomposed into replicas with unit capacity. Examples of matching problems include models with unit traders, additive payoffs, homogeneous goods with multi-unit traders, additively separable payoff functions with heterogeneous commodities, and any model with a combination of these payoff functions. We introduce a new family of payoff functions, called rank-dependent discounts, that nests all of these examples and that is sufficient for decomposability.

Keywords: mechanism design, assignment games, impossibility theorems, component complements, decomposability, rank-dependent discounts

JEL Classifications: C72, D44, D61

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1 Introduction

Two-sided allocation problems abound in economics. Markets assign buyers to sellers, students to colleges, doctors to hospitals, and workers to firms, and enable them to trade at mutually agreeable prices. A fundamental question, first raised and answered in the negative for specific setups by Vickrey (1961), Hurwicz (1973), and Myerson and Satterthwaite (1983), is whether a market maker can costlessly induce buyers and sellers to reveal the information necessary to allocate goods efficiently.

Intuition for the impossibility of ex post efficient trade can be developed by considering the insight from the Vickrey-Clarke-Groves mechanism (VCG) and the literature on dominant strategy implementation that truthful revelation of private information is possible if and only if each agent receives his social marginal product as a transfer (plus a constant). Because the market maker can extract only the social welfare of an allocation but must pay each agent his social marginal products to reveal his information, ex post efficient trade is impossible without running a deficit when the sum of social marginal products exceeds social welfare. In the bilateral trade setup of Myerson and Satterthwaite (1983), this occurs, for example, when the buyer's and the seller's types have overlapping supports. Viewed from this angle, the impossibility result stems from the fact that buyers and sellers are complements in the sense that the sum of their individual social marginal products exceeds their joint social marginal product.¹

Using this insight and a surprising connection between VCG payments and Shapley's (1962) result that in an assignment game agents on opposite sides of the market are complements, we first establish the impossibility of ex post efficient trade for all one-to-one allocations problems with independent private information and identical supports.² This result provides the unifying, and to our knowledge, novel insight that the underlying force behind the impossibility of ex post efficient trade in the two-sided allocation problems of Myerson and Satterthwaite (1983), Gresik and Satterthwaite (1989), and McAfee (1992) is that these are one-to-one allocation problems. The connection between VCG payments and Shapley (1962) also implies that ex

¹As is well-known, the VCG-mechanism is an efficient mechanism that endows agents with dominant strategies. It is due to the independent contributions of Vickrey (1961), Clarke (1971), and Groves (1973). The richness of the environment in Groves (1973) does not permit dominant strategies, but the mechanism developed there endows agents with such strategies in simpler environments such as the ones studied in Groves and Loeb (1975) or here.

²Loertscher, Marx, and Wilkening (2015) invoke Shapley's (1962) theorem to prove the impossibility of trade in one-to-one allocation problems with homogeneous objects. Among other things, we extend this to all matching problems with transfers without any restriction on the nature of the objects that are traded.

post efficient trade with privately informed agents is impossible in the assignment game of Shapley and Shubik (1972), which is popular in the literature on matching with transfers but has received relatively little attention from the mechanism design literature.

Though one-to-one allocation problems provide tractable models and have received considerable interest in the literature on two-sided market design,³ many economically relevant environments involve agents whose types are multi-dimensional and who have demands for or capacities to produce multiple goods. Such environments include the FCC’s upcoming “incentive auction,” the creation of health exchanges under the Affordable Care Act, and two-sided platforms that coordinate production and consumption plans via forward markets when both buyers and sellers can substitute across products. This raises the question as to whether the impossibility of ex post efficient trade can be shown for more general many-to-many allocation problems.

We show that a generalization of complementarity, called **component complements**, implies the impossibility of ex post efficient trade in environments with multi-dimensional types, multi-unit demands, and multi-unit capacities. We say that buyers and sellers are component complements if the social surplus generated in each component of the optimal trading network is less than the sum of the buyers’ and sellers’ individual marginal products within that component.

Of course, the usefulness of the component complements condition depends on the ease with which it can be verified in applications. Our proof of the impossibility of one-to-one allocation problems is based on representing these problems as assignment games and then leveraging the complementarity result of Shapley (1962). A natural question to ask, then, is under what conditions on the primitives — that is, payoff functions — can a general two-sided allocation problem be represented as an assignment game? Calling such a two-sided allocation problem a **matching problem**, we show that the component complements condition is satisfied in any matching problem, which implies the impossibility of ex post efficient trade for all such problems. We then derive a necessary and sufficient condition on agents’ payoff functions — called **decomposability** — for a two-sided allocation problem to be a matching problem.

To be more precise, we call a two-sided allocation problem a matching problem if there exists an assignment game between the objects and replicas of the agents such that a buyer’s payoff in the two-sided allocation problem from consuming a package of objects corresponds to the total

³See Loertscher, Marx, and Wilkening (2015) for a recent survey.

payoff of the replicas of that buyer in the assignment game when the objects in that package are optimally assigned to the buyer's replicas and when a similar condition holds for each seller.⁴ Any optimal matching in the assignment game corresponds to an efficient allocation in the original two-sided allocation problem, and the welfare generated in the two-sided allocation problem and the total payoff generated in the assignment game differ by a constant. This correspondence between the allocation problem and the assignment game carries over to any subset of agents. Extending the results of Shapley (1962) to setups in which *sets* of buyers and sellers are considered, we show that the component complements condition holds in any matching problem, implying that ex post efficient trade is impossible in any matching problem.

Homogeneous objects and additive payoffs are two simple examples of two-sided allocation problems that are matching problems. Although efficient trades are generally not one-to-one with homogeneous objects, multi-unit demands, and multi-unit capacities, one can **decompose** every agent into replicas of himself with unit-demand or unit-supply in such a way that the efficient allocation can be represented as consisting of one-to-one trades between replicas. Section 2 provides an illustrative example of how this works. We show that a two-sided allocation problem is a matching problem if and only if each agent can be decomposed. As decomposability is a condition on each individual agent, it can be verified without having to account for the interaction between different agents. Put differently, impossibility of ex post efficiency obtains in the setups studied by Vickrey (1961), Shapley and Shubik (1972), Myerson and Satterthwaite (1983), Gresik and Satterthwaite (1989), and McAfee (1992) because these are matching models.

In addition to the examples above, two-sided allocation problems that are matching problems include all problems with unit demands and unit supplies, the homogeneous objects model with multi-unit traders and with substitutes utilities and costs, models with additive payoffs, a version of Ausubel's (2006) heterogeneous commodities model with additively separable payoff functions, and any problems involving a mixture of agents with payoff functions of these forms. We unify these models by showing that buyers' and sellers' payoff functions in each of these models exhibit **rank-dependent discounts** and that any model in which agents' payoff functions have this form is a matching problem.

⁴A notionally related but substantively different strand of literature has studied similarities between matching models without a transfers and auctions, which involve transfers, and established equivalences between these two models; see, for example, Kelso and Crawford (1982), Hatfield and Milgrom (2005), or Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp (2015). In contrast, we study under what conditions a two-sided allocation problem is a matching problem, focusing on models with quasi-linear utility and transfers in either case.

This paper combines two strands of literature: matching with transfers and the Bayesian mechanism design literature with two-sided private information. Initiated by Koopmans and Beckmann (1957), Shapley (1962), and Shapley and Shubik (1972) with recent contributions by, among others, Bikhchandani and Ostroy (2002), Echenique, Lee, Shum, and Yenmez (2013), Chambers and Echenique (2015), and Choo (forthcoming), the literature on matching with transfers has, beyond concerns of stability and its relation to the core, paid limited attention to individuals' incentives to reveal what is plausibly their private information. We show that it is impossible to elicit such information without running a deficit if there are least efficient types on both sides of the market that never trade.⁵ From a modeling perspective, our paper extends the package assignment model of Bikhchandani and Ostroy (2002) by introducing heterogeneity on the sellers' side.⁶

As mentioned, the literature on Bayesian mechanism design has predominantly focused on settings with one-dimensional types, with little attention to the connection between allocation problems and matching problems. An important implication of our approach is that a broad class of models are matching problems, including the canonical two-sided market models of Vickrey (1961), Shapley and Shubik (1972), Myerson and Satterthwaite (1983), Gresik and Satterthwaite (1989), McAfee (1992), Loertscher and Mezzetti (2015), and a two-sided, additively separable version of Ausubel's (2006) model with heterogeneous commodities. Rather than being disjoint and independent problems, we demonstrate and characterize the connection between matching models and the models that are predominantly used in Bayesian mechanism design.⁷

In concurrent work, Segal and Whinston (2014) derive impossibility results that revolve around tests for a multi-valued marginal core in allocation games with monetary transfers.

⁵As Shapley and Shubik (1972) show, the core of an assignment game is always nonempty. Furthermore, all core payoffs are efficient. This extends to our environment because each agent is a coalition of replicas in the corresponding assignment game. However, we show that no dominant strategy mechanism exists that achieves ex post efficiency. Thus, we identify an environment in which core payoffs exist but it is not possible to elicit the information required to achieve them.

⁶Among other things, our paper generalizes the conditions under which buyers are known to be substitutes, which is a property that is assumed to hold in parts of the analysis of Bikhchandani and Ostroy (2002).

⁷In a recent paper that takes a bargaining (or robust mechanism design) approach to the problem of bilateral trade with many items, Jackson, Sonnenschein, and Xing (2015) show that whether ex post efficiency is possible depends on whether the two agents bargain over all items at once or independently. The difference between the possibility results that Jackson, Sonnenschein, and Xing (2015) obtain and the impossibility results derived here stems from a difference regarding the following assumptions: in line with Myerson and Satterthwaite (1983), we assume that for every type realization of the other agent(s), an agent may be of such an unproductive type that it is efficient that he does not trade at all, while the possibility results of Jackson, Sonnenschein, and Xing (2015) are obtained under the assumption that the bilateral surplus over all items is positive for all type realizations for which the density is positive.

Our component complements condition is a strengthening of the marginal core condition. The marginal core is therefore useful in providing general conditions under which the impossibility results hold but, as with our component complements condition, may be difficult to apply in practice. Our results regarding matching problems provide a complementary approach for testing for the impossibility of ex post efficient trade. Importantly, using our decomposability results, these conditions can be checked at the level of individuals' payoff functions.

Our paper also relates to the literature on the micro-foundations of the canonical model of price formation in markets. As first noticed by Arrow (1959), the Walrasian model is silent about the institutions that simultaneously discover and set the market clearing prices. Recent contributions by Satterthwaite, Williams, and Zachariadis (forthcoming) and Satterthwaite, Williams, and Zachariadis (2015) have focused on the performance of the k -double-auction in environments with unit traders, allowing for the possibility of correlated types and interdependent values. Our work is complementary to this research agenda. We do not restrict the mechanism that the market maker employs, other than imposing incentive and individual rationality constraints, and we allow for a general trading environment apart from imposing private values (and independence for the Bayesian version of the impossibility result in Appendix B).

The remainder of this paper is organized as follows. Section 2 provides an illustrative example. Section 3 describes the setup. In Section 4, we formally introduce the component complements condition and derive the impossibility result for dominant strategy mechanisms. We extend this result to Bayesian incentive compatible, interim individually rational mechanisms in Appendix B. Section 5 introduces assignment games and proves an impossibility result for one-to-one allocation problems. Section 6 defines matching problems and shows that our component complements condition is satisfied in matching problems. Section 7 provides a necessary and sufficient condition, decomposability, for a two-sided allocation problem to be a matching problem and defines a general class of models, those where agents' payoff functions exhibit rank-dependent discounts, in which ex post efficient trade is impossible. Section 8 concludes. Proofs for the results in Sections 5, 6 and 7 are in Appendix A.

2 Example

A novel insight in this paper is that some two-sided allocation problems — those that we refer to as matching problems — can be represented by an assignment game between replicas of the agent and objects. This section provides an example of such a representation and highlights

some of the properties of the assignment game that are useful in proving our impossibility results.

Consider a two-sided allocation problem with two buyers, called David and Martin, and a seller, called Lloyd. Lloyd can produce two heterogeneous objects, A and B , at cost 2 for A , 3 for B , and 6 for producing both A and B . David values A at 9, B at 5, and the package AB at 12, while Martin's values are 5, 6, and 10, respectively.

With both buyers present, the efficient allocation assigns object A to David and object B to Martin and generates a total welfare of 9. Because the efficient allocation involves Lloyd trading with both David and Martin, this two-sided allocation problem cannot be represented as one-to-one trades between David, Martin, and Lloyd. However, one can create replicas of agents in such a way that a corresponding assignment game between the objects and the replicas of the agents exists. Panel (a) of Table 1 defines the assignment game that is equivalent to the two-sided allocation problem just described. For David and Martin, the first replica in Table 1 contains the marginal utility from receiving either A or B , while the second replica represents the marginal utility from adding either A or B to the package that already includes the other object. For Lloyd, replica 2 in the assignment game represents the marginal cost of producing either A or B , while replica 1 records the incremental cost of producing either A or B given that the other one is produced.

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Table 1: An assignment game corresponding to the two-sided allocation problem between David, Martin, and Lloyd.

One can view the assignment of an object to a buyer as meaning that the buyer receives the object, while assigning the object to a seller can be viewed as meaning that the seller does *not* have to produce the object. Thus, if no replica of Lloyd receives an object, the interpretation is that Lloyd produces both objects. It is immaterial which object is given to which replica of a buyer if that buyer receives both objects because the diagonal entries always sum to 12 and 10, respectively, and likewise for the seller. However, if any agent receives only one object,

efficiency dictates that his first replica be assigned the object.

The solution to this assignment game, shown in bold in Panel (a) of Table 1, assigns object A to the first replica of David and B to the first replica of Martin. Because no replica of Lloyd receives an object, Lloyd produces both objects. The total payoff from this assignment is 15 and differs from the total welfare of 9 in the efficient allocation by 6, which is the cost imposed on Lloyd associated with producing both objects.

An important feature of the assignment game constructed above is that the correspondence between the allocation problem and the assignment game carries over to any subset of agents. To see this, it is useful to compare the cases in which David and Martin, respectively, are removed from the model. If David is not present in the original allocation problem, efficiency dictates that Lloyd continue to produce both objects and that they be given to Martin, generating a total welfare of 4. In the corresponding assignment game with both replicas of David removed, shown in Panel (b) of Table 1, both objects are optimally assigned to replicas of Martin and the total value from these assignments is 10. As in the original case with all agents included, the difference is 6. Likewise, if Martin is removed from the original two-sided allocation problem, only object A is traded under efficiency, generating a welfare of 7. In the corresponding assignment game, shown in Panel (c) of Table 1, object A is assigned to replica 1 of David and object B is assigned to replica 1 of Lloyd, generating a value in the assignment game of 13. The difference is again 6. Thus, the correspondence between the solution to the two-sided allocation problem and the assignment game holds when either buyer is removed.

If Lloyd is removed, no trade occurs and a welfare of 0 is generated. The corresponding assignment game is obtained by removing the replicas of Lloyd as well as the objects he is able to produce. In this simple example, an empty matrix is obtained yielding a value of 0. The difference between the welfare in the allocation problem and the value in the assignment game is 0 because no object can be produced.

As will be seen in the rest of the paper, the component complements condition is satisfied if, within each component, the sum of the individual marginal surpluses is greater than the total surplus generated in that component. For the values above, where all agents belong to the same component, the component complements condition holds since the marginal surplus of David is 5 ($=9-4$), the marginal surplus of Martin is 2 ($=9-7$), the marginal surplus of Lloyd is 9, and the total surplus generated in the component is 9. In the assignment game, a similar relationship holds as the replicas of buyers and sellers are removed; the difference in the output

of the assignment game between the full game and the game without David is 5 ($=15-10$). Likewise the difference in output is 2 when Martin is removed ($=15-13$) and 15 when Lloyd is removed ($=15-0$). As the total output of the assignment game is 15 ($=9+6$), there is a direct correspondence between the marginal surpluses in the allocation game and the assignment game.

Under VCG, Lloyd would receive a payment of 15 while David and Martin would each pay 4. The net deficit is 7, which corresponds to the difference between the sum of the marginal surpluses and the surplus generated in the component. As this net difference can also be calculated by removing sets of replicas from the assignment game, we can use properties of the assignment game to determine conditions for which the impossibility theorem holds. We use this intuition to generalize the impossibility theorem to matching problems in Section 6.2.

While the example above highlights how two-sided allocation problems can be represented as an assignment games even in the presence of heterogeneous objects, not all two-sided allocation problems can be represented in this way. For example, if David values the package AB at 15, there is no way of representing the problem as an assignment game.

Intuition suggests that the dividing line between agents that are and are not decomposable is whether they see objects as substitutes or complements. While Proposition 3 shows that objects being substitutes is a necessary condition for decomposability, it is not sufficient. As one example, consider an extension of our model where there is a third object, called C , and assume that David derives a stand-alone payoff of 7 for this object, a payoff of 14 for the package AC , and a payoff of 11 for the package BC . David is not decomposable and the two-sided allocation problem is not a matching problem, irrespective of the payoff functions of other agents.

In Section 7.2, we show that an agent is decomposable if his payoff function exhibits rank-dependent discounts. To the best of our knowledge, none of the existing definitions of substitutability that have been used in the various literatures are sufficient (see, e.g., Hatfield, Immorlica, and Kominers (2012) or Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp (2015)).

3 Setup

A two-sided allocation problem consists of B buyers $b \in \mathbb{B}$, S sellers $s \in \mathbb{S}$, and O objects $o \in \mathbb{O}$. Each seller has O_s objects $o_s \in \mathbb{O}_s$ that only he can produce. Each object can be produced by one seller. Therefore, $\{\mathbb{O}_s\}_{s \in \mathbb{S}}$ is a partition of \mathbb{O} . We call \mathbb{O}_s the **potential set** of seller s .

Each buyer b can consume any package $x_b \subseteq \mathbb{O}$. Let $\mathbb{P} := \mathcal{P}(\mathbb{O})$ be the set of all possible packages that can be consumed and define $P = 2^{\mathbb{O}}$ to be the cardinality of the set.⁸ Each seller s can produce any package $x_s \subseteq \mathbb{O}_s$. Let $\mathbb{P}_s := \mathcal{P}(\mathbb{O}_s)$ and define its cardinality as $P_s = 2^{|\mathbb{O}_s|}$.

An allocation $x = ((x_b)_{b \in \mathbb{B}}, (x_s)_{s \in \mathbb{S}})$ corresponds to trades between buyers and sellers, where x_b is the package b receives and x_s is the package s produces. An allocation is **feasible** if (i) each seller produces a subset of his potential set: $x_s \subseteq \mathbb{O}_s$ for all $s \in \mathbb{S}$, (ii) each object that is consumed has been produced: $x_b \subseteq \cup_{s \in \mathbb{S}} x_s$ for all $b \in \mathbb{B}$, and (iii) each object is consumed at most once: $x_b \cap x_{b'} = \emptyset$ for any $b, b' \in \mathbb{B}$ with $b \neq b'$. Let X denote the set of feasible allocations.

Buyer b 's payoff function takes the quasi-linear form $u_b(x_b, \mathbf{v}_b) - t_b$, where x_b is the package the buyer receives, \mathbf{v}_b is buyer b 's type, and t_b is a monetary transfer made by buyer b to the mechanism. Similarly, seller s 's payoff function is $t_s - \kappa_s(x_s, \mathbf{c}_s)$, where t_s is a monetary transfer made by the mechanism to seller s . We normalize the gross surplus of non-trading players to zero, i.e., $u_b(\emptyset, \cdot) = 0$ and $\kappa_s(\emptyset, \cdot) = 0$. We also assume that $u_b(x_b, \mathbf{v}_b)$ and $\kappa_s(x_s, \mathbf{c}_s)$ only depend on the agents' own allocations and on their own private types.⁹ We allow for free disposal by both buyers and sellers. For a buyer of type \mathbf{v}_b who is allocated package x_b , define the **upper envelope function** \bar{u}_b :

$$\bar{u}_b(x_b, \mathbf{v}_b) := \max_{\hat{x}_b} u_b(\hat{x}_b, \mathbf{v}_b) \quad \text{such that} \quad \hat{x}_b \subseteq x_b. \quad (1)$$

The upper envelope function returns the largest possible utility from the available packages that can be constructed from objects in x_b allowing for free disposal. By construction, it is monotonically increasing.

Analogously, for a seller of type \mathbf{c}_s who is allocated package x_s , define the **lower envelope function** $\underline{\kappa}_s$:

$$\underline{\kappa}_s(x_s, \mathbf{c}_s) := \min_{\hat{x}_s} \kappa_s(\hat{x}_s, \mathbf{c}_s) \quad \text{such that} \quad x_s \subseteq \hat{x}_s \subseteq \mathbb{P}_s. \quad (2)$$

The lowest envelope function represents the cost for s to produce any package that includes objects in x_s , accounting for cases in which it may be cheaper for s to produce a larger package and to dispose of some of the objects produced. By construction, the lower envelope function is also monotonically increasing.

⁸For instance, if there are two objects a and b , the set of possible packages contains $\{\emptyset\}$, $\{a\}$, $\{b\}$, and $\{a, b\}$. The cardinality of the set is $2^2 = 4$.

⁹For the purposes of Theorem 1, we only need to assume that there are no negative externalities for non-trading agents.

Types are agents' own private information and are defined over the packages that they produce or consume. This implies that a buyer b 's type $\mathbf{v}_b \in \mathbb{R}^P$ is a P -dimensional vector of values corresponding to each possible consumption package. A seller s 's type $\mathbf{c}_s \in \mathbb{R}^{P_s}$ is a P_s -dimensional vector of costs corresponding to each possible production package. Let V_b denote the set of possible types for buyer b and C_s denote the set of possible types for seller s . The sets $V = \prod_{b \in \mathbb{B}} V_b$ and $C = \prod_{s \in \mathbb{S}} C_s$ are the products of the sets of types, with typical elements \mathbf{v} and \mathbf{c} . We assume that V_b and C_s are closed and convex sets. Later we impose additional restrictions on the type space $V \times C$.

For a feasible allocation $x \in X$ and types (\mathbf{v}, \mathbf{c}) , define **social welfare** as

$$W(x, \mathbf{v}, \mathbf{c}) = \sum_{b \in \mathbb{B}} \bar{u}_b(x_b, \mathbf{v}_b) - \sum_{s \in \mathbb{S}} \underline{\kappa}_s(x_s, \mathbf{c}_s). \quad (3)$$

Denote maximum welfare by $W^*(\mathbf{v}, \mathbf{c}) := W(x^*, \mathbf{v}, \mathbf{c})$, where x^* denotes an element of the set of maximizers. Because we have assumed free disposal, restricting attention to upper envelope utility functions and lower envelope cost functions is without loss of generality for the purpose of calculating the maximal welfare. When there is no risk of confusion, we drop the dependence of W^* on (\mathbf{v}, \mathbf{c}) .

It is also useful to denote social welfare when a subset of buyers $\mathbb{I} \subseteq \mathbb{B}$ and a subset of sellers $\mathbb{J} \subseteq \mathbb{S}$ are excluded. For any feasible allocation x of that allocation problem, social welfare is

$$W_{-\mathbb{I}, -\mathbb{J}}(x, \mathbf{v}, \mathbf{c}) = \sum_{b \in \mathbb{B} \setminus \mathbb{I}} \bar{u}_b(x_b, \mathbf{v}_b) - \sum_{s \in \mathbb{S} \setminus \mathbb{J}} \underline{\kappa}_s(x_s, \mathbf{c}_s). \quad (4)$$

The maximum welfare of this smaller two-sided allocation problem is denoted $W_{-\mathbb{I}, -\mathbb{J}}^*(\mathbf{v}, \mathbf{c})$. We drop the dependency on (\mathbf{v}, \mathbf{c}) when there is no risk of confusion and write $W_{-\mathbb{I}, -\mathbb{J}}^*$.

A **direct mechanism** is a triple $(\chi, \mathbf{t}^\beta, \mathbf{t}^\sigma)$, where $\chi : V \times C \rightarrow X$ is an **allocation rule** and $\mathbf{t}^\beta : V \times C \rightarrow \mathbb{R}^B$ and $\mathbf{t}^\sigma : V \times C \rightarrow \mathbb{R}^S$ are the **payment rules**. Thus, given reports (\mathbf{v}, \mathbf{c}) , $\chi(\mathbf{v}, \mathbf{c})$ is the chosen allocation, buyer b pays $t_b^\beta(\mathbf{v}, \mathbf{c})$, and seller s receives $t_s^\sigma(\mathbf{v}, \mathbf{c})$. An allocation rule is **ex post efficient** if it specifies an efficient allocation for every (\mathbf{v}, \mathbf{c}) . For the purpose of deriving the conditions under which ex post efficiency is impossible without running a deficit, the well-known revelation principle (Myerson, 1981) implies that we can restrict attention to direct mechanisms without loss of generality.

Any allocation of objects from sellers to buyers generates a set of trade **links** between buyers and sellers that can be represented as a bipartite graph. A link between b and s is denoted $\lambda_{b,s}(x) \in \{0, 1\}$, where $\lambda_{b,s}(x) = 1$ means that seller s produces a non-empty package of objects for buyer b and $\lambda_{b,s}(x) = 0$ means that seller s produces nothing for buyer b .

The trade links generated by an allocation can be partitioned into **components**, where component G consists of a selection of buyers and sellers that are only linked to other buyers and sellers in the same component. Let $\mathbb{G}(x)$ be the set of all components associated with allocation x . Denote by $\mathbb{B}(G)$ and $\mathbb{S}(G)$ the sets of buyers and sellers in component G .

We refer to a problem as a **one-to-one allocation problem** if for all type profiles (\mathbf{v}, \mathbf{c}) and efficient allocations x^* the following two conditions hold:

- (i) $\lambda_{b,s}(x^*) = 1$ for some s implies $\lambda_{b,s'}(x^*) = 0$ for all $s' \neq s$ for all $b \in \mathbb{B}$
- (ii) $\lambda_{b,s}(x^*) = 1$ for some b implies $\lambda_{b',s}(x^*) = 0$ for all $b' \neq b$ for all $s \in \mathbb{S}$.

Any other setup is referred to as a **many-to-many allocation problem**. Many-to-many matching problems include setups in which one buyer is matched with multiple sellers or one seller is matched with multiple buyers.

We now impose additional structure on the type space $V \times C$. Let \mathbf{v}_{-b} be the vector of types of buyers other than b and let $(\hat{\mathbf{v}}_b, \mathbf{v}_{-b})$ be the vector \mathbf{v} with the type of buyer b replaced by $\hat{\mathbf{v}}_b$, and similarly for sellers. We assume that for every $b \in \mathbb{B}$, there exists a **least efficient** type $\underline{\mathbf{v}}_b \in V_b$ such that for any type profile $(\underline{\mathbf{v}}_b, \mathbf{v}_{-b}, \mathbf{c})$ and at every efficient allocation $x^*(\underline{\mathbf{v}}_b, \mathbf{v}_{-b}, \mathbf{c})$, $x_b = \emptyset$. Analogously, we assume that for every $s \in \mathbb{S}$, there exists a least efficient type $\bar{\mathbf{c}}_s \in C_s$ such that for any type profile $(\mathbf{v}, \bar{\mathbf{c}}_s, \mathbf{c}_{-s})$ and efficient allocation $x^*(\mathbf{v}, \bar{\mathbf{c}}_s, \mathbf{c}_{-s})$, $x_s = \emptyset$.

Following Holmström (1979), we say the type space is **smoothly connected** if for all buyers b , the set of mappings $\{u_b(\cdot, \mathbf{v}_b) : X \rightarrow \mathbb{R} \mid \mathbf{v}_b \in V_b\}$ is smoothly connected, and for all sellers s , the set of mappings $\{\kappa_s(\cdot, \mathbf{c}_s) : X \rightarrow \mathbb{R} \mid \mathbf{c}_s \in C_s\}$ is smoothly connected (for a formal definition of smooth connectedness, see Definition 1 in Holmström (1979)). We assume that the type space is smoothly connected.

4 Complementarity and the Impossibility of Ex Post Efficiency

In this section, we introduce and discuss the component complements condition, which says that the surplus generated within a component is less than or equal to the sum of the marginal contributions of buyers and sellers in the same component, and show that this is sufficient for the VCG mechanism never to run a budget surplus. We then invoke a revenue equivalence theorem to conclude that any other dominant strategy mechanism that respects individual rationality never runs a budget surplus and runs a deficit under a strict version of the condition.

4.1 Complements and Component Complements

Buyers and sellers are **complements** if for all (\mathbf{v}, \mathbf{c}) and $(b, s) \in \mathbb{B} \times \mathbb{S}$,

$$W_{-b, \cdot}^* - W_{-b, -s}^* + W_{\cdot, -s}^* - W_{-b, -s}^* \leq W^* - W_{-b, -s}^*.$$

The expressions $W_{-b, \cdot}^* - W_{-b, -s}^*$ and $W_{\cdot, -s}^* - W_{-b, -s}^*$ capture, respectively, the individual marginal contribution to welfare of buyer b and seller s to an economy that consists of all buyers other than b and all sellers other than s . The right side is the marginal contribution of adding the pair consisting of b and s to the economy without this pair. Thus, the inequality simply states that the marginal contribution of the pair is not less than the sum of the individual marginal contributions when the other agent is not there. In this sense, buyers and sellers are complements. An equivalent representation of the complements condition can be found by adding W^* to both sides of this equation and rearranging:

$$W^* - W_{-b, \cdot}^* + W^* - W_{\cdot, -s}^* \geq W^* - W_{-b, -s}^*. \quad (\text{C})$$

We use this alternative representation in the rest of the paper. It says that the sum of the individual marginal products when the other agent is there exceeds the marginal product of the pair.

Buyers and sellers are **component complements** if for all (\mathbf{v}, \mathbf{c}) , $(b, s) \in \mathbb{B} \times \mathbb{S}$, and $G \in \mathbb{G}(x^*)$,

$$\sum_{b \in \mathbb{B}(G)} [W^* - W_{-b, \cdot}^*] + \sum_{s \in \mathbb{S}(G)} [W^* - W_{\cdot, -s}^*] \geq W^* - W_{-\mathbb{B}(G), -\mathbb{S}(G)}^*. \quad (\text{CC})$$

(CC) requires that the individual marginal contributions to welfare of all buyers and sellers in a component exceeds the surplus of the component.¹⁰

¹⁰Buyers and sellers satisfy the **marginal core** condition of Segal and Whinston (2014) if and only if

$$\sum_{b \in \mathbb{B}} [W^* - W_{-b, \cdot}^*] + \sum_{s \in \mathbb{S}} [W^* - W_{\cdot, -s}^*] \geq W^*.$$

Because buyers and sellers who do not trade produce no surplus, this condition can be rewritten as

$$\sum_{G \in \mathbb{G}} \left[\sum_{b \in \mathbb{B}(G)} [W^* - W_{-b, \cdot}^*] + \sum_{s \in \mathbb{S}(G)} [W^* - W_{\cdot, -s}^*] \right] > \sum_{G \in \mathbb{G}} \left[W^* - W_{-\mathbb{B}(G), -\mathbb{S}(G)}^* \right].$$

As can be seen by comparing this equation to (CC), the marginal core condition is satisfied any time (CC) holds. We use the component complements rather than the marginal core condition because the main intuition and results of this paper come from the complementarity between buyers and sellers that exist at the component level.

We say that buyers and sellers are **strict component complements** if (CC) holds and if there exist a type profile (\mathbf{v}, \mathbf{c}) and component G such that (CC) holds strictly. In the one-to-one setting, non-empty components are always of size two and so (CC) is satisfied if buyers and sellers are complements. That is, for one-to-one allocation problems (C) implies (CC).

4.2 Dominant Strategy Mechanisms

A mechanism $(\chi, \mathbf{t}^\beta, \mathbf{t}^\sigma)$ is **dominant strategy incentive compatible (DIC)** if for each buyer b , type profile (\mathbf{v}, \mathbf{c}) , and type $\hat{\mathbf{v}}_b$ for buyer b ,

$$\bar{u}_b(\chi_b(\mathbf{v}, \mathbf{c}), \mathbf{v}_b) - t_b^\beta(\mathbf{v}, \mathbf{c}) \geq \bar{u}_b(\chi_b(\hat{\mathbf{v}}_b, \mathbf{v}_{-b}, \mathbf{c}), \mathbf{v}_b) - t_b^\beta(\hat{\mathbf{v}}_b, \mathbf{v}_{-b}, \mathbf{c}),$$

and for each seller s , type profile (\mathbf{v}, \mathbf{c}) , and type $\hat{\mathbf{c}}_s$ for seller s ,

$$t_s^\sigma(\mathbf{v}, \mathbf{c}) - \underline{\kappa}_s(\chi_s(\mathbf{v}, \mathbf{c}), \mathbf{c}_s) \geq t_s^\sigma(\mathbf{v}, \hat{\mathbf{c}}_s, \mathbf{c}_{-s}) - \underline{\kappa}_s(\chi_s(\mathbf{v}, \hat{\mathbf{c}}_s, \mathbf{c}_{-s}), \mathbf{c}_s),$$

where χ_b and χ_s denote the packages that buyer b receives and seller s produces under allocation rule χ .

The mechanism satisfies **ex post individual rationality (EIR)** if for each buyer b and type profile (\mathbf{v}, \mathbf{c}) , $\bar{u}_b(\chi_b(\mathbf{v}, \mathbf{c}), \mathbf{v}_b) - t_b^\beta(\mathbf{v}, \mathbf{c}) \geq 0$ and for each seller s and type profile (\mathbf{v}, \mathbf{c}) , $t_s^\sigma(\mathbf{v}, \mathbf{c}) - \underline{\kappa}_s(\chi_s(\mathbf{v}, \mathbf{c}), \mathbf{c}_s) \geq 0$.

In what follows, we focus on the Vickrey-Clarke-Groves (VCG) mechanism, which is a DIC, EIR direct mechanism that uses an ex post efficient allocation rule.¹¹ Let $x^*(\mathbf{v}_b, \mathbf{v}_{-b}, \mathbf{c}) \in X(\mathbf{v}_b, \mathbf{v}_{-b}, \mathbf{c})$ and define for every buyer b ,

$$W_{-b}^{VCG} := \min_{\mathbf{v}_b \in V_b} W(x^*(\mathbf{v}_b, \mathbf{v}_{-b}, \mathbf{c}), \mathbf{v}_b, \mathbf{v}_{-b}, \mathbf{c}). \quad (5)$$

Similarly, for every seller s , define

$$W_{-s}^{VCG} := \min_{\mathbf{c}_s \in C_s} W(x^*(\mathbf{v}, \mathbf{c}_s, \mathbf{c}_{-s}), \mathbf{v}, \mathbf{c}_s, \mathbf{c}_{-s}). \quad (6)$$

¹¹While the VCG mechanism is named after the independent contributions by Vickrey (1961), Clarke (1971), and Groves (1973), the genealogy of the term is interesting and seems, to us, still somewhat unclear. To the best of our knowledge, Makowski and Ostroy (1987) were the first to use it. Green and Laffont (1977) seem to have coined the term Groves schemes in honor of the seminal contribution by Groves (1973), which specializes to VCG with an appropriately chosen constant. Mas-Colell, Whinston, and Green (1995, Section 23.C) refer to what we call VCG as a Groves-Clarke scheme. Tracing out the origins of the mechanism is not only complicated by the fact that it has not been introduced by anyone deliberately (as opposed to, e.g., Nash equilibrium), but also by the circumstance that the environment for which Groves schemes were first introduced (Groves, 1973) was too rich to permit dominant strategies. Only when simplified to the setup studied in Groves and Loeb (1975) does the mechanism endow agents with dominant strategies. More importantly, there appear to be conflicting definitions of VCG that coincide when there exist least efficient types but differ when such types do not exist.

Notice that W_{-b}^{VCG} and W_{-s}^{VCG} are independent of, respectively, \mathbf{v}_b and \mathbf{c}_s .

The VCG mechanism selects $\chi(\mathbf{v}, \mathbf{c}) \in X^*$, requires each buyer to pay a transfer payment of

$$t_b^\beta(\mathbf{v}, \mathbf{c}) = W_{-b}^{VCG} - (W^* - \bar{u}_b(\chi(\mathbf{v}, \mathbf{c}), \mathbf{v}_b)),$$

and pays each seller a transfer payment of

$$t_s^\sigma(\mathbf{v}, \mathbf{c}) = W^* + \underline{\kappa}_s(\chi(\mathbf{v}, \mathbf{c}), \mathbf{c}_s) - W_{-s}^{VCG}.$$

The VCG mechanism is DIC because it aligns every agent's incentives with those of society by making the objective function of every individual equal to social welfare plus a constant (W_{-s}^{VCG} for a seller and $-W_{-b}^{VCG}$ for a buyer). It is also ex post individually rational because $\bar{u}_b(x^*, \mathbf{v}_b) - t_b^\beta(\mathbf{v}, \mathbf{c}) \geq 0$ and $t_s^\sigma(\mathbf{v}, \mathbf{c}) - \underline{\kappa}_s(x^*, \mathbf{c}_s) \geq 0$ for any b, s , and (\mathbf{v}, \mathbf{c}) . By the revenue equivalence arguments of Green and Laffont (1977) and Holmström (1979), the VCG mechanism is the revenue maximizing mechanism among all mechanisms that respect agents' individual rationality constraints ex post and endow them with dominant strategies. Our focus on VCG is therefore without loss of generality.

Because we have assumed that there exists a least efficient buyer type, $\underline{\mathbf{v}}_b$ is the minimizer of $W(x^*(\mathbf{v}_b, \mathbf{v}_{-b}, \mathbf{c}), \mathbf{v}_b, \mathbf{v}_{-b}, \mathbf{c})$, and similarly $\bar{\mathbf{c}}_s$ is the minimizer of $W(x^*(\mathbf{v}, \mathbf{c}_s, \mathbf{c}_{-s}), \mathbf{v}, \mathbf{c}_s, \mathbf{c}_{-s})$. Thus, in our environment, $W_{-b}^{VCG} = W_{-b}^*$ and $W_{-s}^{VCG} = W_{-,s}^*$.¹² Revenue to the mechanism when the types are (\mathbf{v}, \mathbf{c}) , denoted $R(\mathbf{v}, \mathbf{c})$, is thus

$$R(\mathbf{v}, \mathbf{c}) = \sum_{b \in \mathbb{B}} t_b^\beta(\mathbf{v}, \mathbf{c}) - \sum_{s \in \mathbb{S}} t_s^\sigma(\mathbf{v}, \mathbf{c}) = W^* + \sum_{b \in \mathbb{B}} [W_{-b}^* - W^*] + \sum_{s \in \mathbb{S}} [W_{-,s}^* - W^*]. \quad (7)$$

Using (7), the condition for VCG revenue, $R(\mathbf{v}, \mathbf{c})$, to be less than or equal to zero is equivalent to

$$\sum_{b \in \mathbb{B}} [W^*(\mathbf{v}, \mathbf{c}) - W_{-b}^*(\mathbf{v}, \mathbf{c})] + \sum_{s \in \mathbb{S}} [W^*(\mathbf{v}, \mathbf{c}) - W_{-,s}^*(\mathbf{v}, \mathbf{c})] \geq W^*(\mathbf{v}, \mathbf{c}). \quad (8)$$

Because buyers and sellers in positive-surplus components are the only buyers and sellers to have positive surplus, the left side can be rewritten as:

$$\sum_{G \in \mathbb{G}(x^*)} \left[\sum_{b \in \mathbb{B}(G)} [W^*(\mathbf{v}, \mathbf{c}) - W_{-b}^*(\mathbf{v}, \mathbf{c})] + \sum_{s \in \mathbb{S}(G)} [W^*(\mathbf{v}, \mathbf{c}) - W_{-,s}^*(\mathbf{v}, \mathbf{c})] \right] \geq W^*(\mathbf{v}, \mathbf{c}). \quad (9)$$

¹²Many authors define VCG with W_{-b}^* and $W_{-,s}^*$ directly. For example Green and Laffont (1977), Holmström (1979), Makowski and Ostroy (1987), Mas-Colell, Whinston, and Green (1995), and Milgrom (2004) use this definition. Authors using our definition include Krishna (2002) (which is based on Krishna and Perry (2000)), Segal and Whinston (2014), Segal and Whinston (2011), and Loertscher, Marx, and Wilkening (2015). The generalization of VCG to interdependent values by Ausubel (1999) also makes use of the more general definition.

(CC) implies:

$$\begin{aligned}
& \sum_{G \in \mathbb{G}(x^*)} \left[\sum_{b \in \mathbb{B}(G)} [W^*(\mathbf{v}, \mathbf{c}) - W_{-b, \cdot}^*(\mathbf{v}, \mathbf{c})] + \sum_{s \in \mathbb{S}(G)} [W^*(\mathbf{v}, \mathbf{c}) - W_{\cdot, -s}^*(\mathbf{v}, \mathbf{c})] \right] \\
& \geq \sum_{G \in \mathbb{G}(x^*)} \left[W^* - W_{-\mathbb{B}(G), -\mathbb{S}(G)}^* \right] \\
& = W^*(\mathbf{v}, \mathbf{c}).
\end{aligned}$$

Thus, if (CC) holds, (8) holds.

We say that **ex post efficient trade is impossible** for a two-sided allocation problem if there is no ex post efficient DIC-EIR mechanism for that allocation problem that never runs a deficit, i.e. has revenue $R(\mathbf{v}, \mathbf{c}) < 0$ for an open set of type profiles and never runs a budget surplus for any type profile. It is clear from the argument above that for a given (\mathbf{v}, \mathbf{c}) , if (CC) holds strictly for at least one component, then VCG revenue is negative, i.e., $R(\mathbf{v}, \mathbf{c}) < 0$. To see that (CC) holds strictly for an open set of type profiles, consider the type profile $\mathbf{v}_{-b} = \underline{\mathbf{v}}_{-b}$ and $\mathbf{c}_{-s} = \bar{\mathbf{c}}_{-s}$ and \mathbf{v}_b and \mathbf{c}_s such that s and b efficiently trade with each other and no other trades occur. Because the type-space is smoothly connected, s and b will still efficiently trade with each other while all the other agents will efficiently not trade in an open neighborhood of these types. Because b and s remain the only traders, the sum of their marginal products strictly exceeds social welfare. Thus, we have the following result:

Theorem 1 *Ex post efficient trade is impossible in any allocation problem in which (CC) holds.*

5 Impossibility in One-to-One Allocation Problems

In this section, we analyze one-to-one allocation problems. These are natural when agents' types are one-dimensional and buyers have unit demand and sellers have unit capacities and objects are homogeneous. However, one-to-one allocation problems are not confined to setups with one-dimensional types. For example, the housing market model of Shapley and Shubik (1972) has multi-dimensional types because buyers with unit demand have heterogeneous values for the sellers' houses.

In what is referred to in the literature as an assignment game, Shapley (1962) derives a result that implies that buyers and sellers are complements. In this section, we show that one-to-one allocation problems can always be represented as an assignment game. We use this result to prove that (CC) holds in all one-to-one allocation problems.

An **assignment game** is defined by a $B \times S$ -dimensional **payoff matrix**

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,S} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,S} \\ \vdots & \vdots & \ddots & \vdots \\ a_{B,1} & a_{B,2} & \cdots & a_{B,S} \end{pmatrix} \quad (10)$$

that contains all the payoff relevant information in the game. The assignment game uses a corresponding $B \times S$ -dimensional **matching matrix** L with elements $l_{b,s} \in \{0, 1\}$, where $l_{b,s} = 1$ if b and s are matched together and $l_{b,s} = 0$ if they are not. A matching is **feasible** if the sum of any row or column in the matching matrix is at most one. A feasible match is **efficient** if the value of the match

$$\sum_{b \in \mathbb{B}} \sum_{s \in \mathbb{S}} l_{b,s} a_{b,s} \quad (11)$$

is maximized. As is well-known, the efficient match from an assignment game can be found by solving the following linear programming problem:¹³

$$\begin{aligned} & \max_L \sum_{b \in \mathbb{B}} \sum_{s \in \mathbb{S}} l_{b,s} a_{b,s} \\ & \text{subject to} \quad \sum_{s \in \mathbb{S}} l_{b,s} \leq 1 \quad \text{for all } b \in \mathbb{B} \\ & \text{and} \quad \sum_{b \in \mathbb{B}} l_{b,s} \leq 1 \quad \text{for all } s \in \mathbb{S}. \end{aligned} \quad (12)$$

We denote the value generated from an efficient match by $V^*(A)$. Similarly, we let $V^*(A_{-\mathbb{I}, -\mathbb{J}})$ be the maximal value of an assignment game when rows $\mathbb{I} \subseteq \mathbb{B}$ and columns $\mathbb{J} \subseteq \mathbb{S}$ are removed.

Two buyers (or two sellers) are **substitutes** for each other if their joint marginal payoff exceeds the sum of their individual marginal payoffs:

$$\begin{aligned} V^*(A) - V^*(A_{-b, \cdot}) + V^*(A) - V^*(A_{-b', \cdot}) &\leq V^*(A) - V^*(A_{-\{b, b'\}, \cdot}) \quad \text{for any } b, b' \in \mathbb{B} \\ V^*(A) - V^*(A_{-\cdot, s}) + V^*(A) - V^*(A_{-\cdot, s'}) &\leq V^*(A) - V^*(A_{-\cdot, \{s, s'\}}) \quad \text{for any } s, s' \in \mathbb{S}. \end{aligned}$$

Analogously, a buyer and a seller are **complements** to one another if their joint marginal payoff is smaller than the sum of their individual marginal payoffs. That is, for any $b \in \mathbb{B}$ and $s \in \mathbb{S}$,

$$V^*(A) - V^*(A_{-b, \cdot}) + V^*(A) - V^*(A_{-\cdot, s}) \geq V^*(A) - V^*(A_{-b, -s}). \quad (13)$$

Shapley (1962) establishes the following result:

¹³Solutions to this problem can be found by inspection if A is small enough, by using the Hungarian algorithm of Kuhn (1955), or through linear programming techniques (Dantzig, 1963).

Theorem 2 (Shapley, 1962, Theorems 1 and 2) *In any assignment game, any two agents on the same side of the market are substitutes for each other while any two agents on opposite sides are complements to each other.*

For any allocation problem in which the efficient allocation is one-to-one for all (\mathbf{v}, \mathbf{c}) , a buyer b and a seller s who are matched always trade the package that maximizes their joint surplus, whose value we denote by $a_{b,s} := \max \{ \max_{x \in \mathbb{P}_s} \bar{u}_b(x, \mathbf{v}_b) - \underline{k}_s(x, \mathbf{c}_s), 0 \}$. Using these elements to construct an assignment game payoff matrix A as in (10), it follows that the one-to-one allocation problem is equivalent to an assignment game.

At an optimal matching in the resulting assignment game, $W^* - W_{-b,.}^* = V^*(A) - V^*(A_{-b,.})$, $W^* - W_{.,-s}^* = V^*(A) - V^*(A_{.,-s})$ and $W^* - W_{-b,-s}^* = V^*(A) - V^*(A_{-b,-s})$. By Theorem 2, buyers and sellers in the assignment game are complements. This implies that for any $b \in \mathbb{B}$ and $s \in \mathbb{S}$,

$$V^*(A) - V^*(A_{-b,.}) + V^*(A) - V^*(A_{.,-s}) \geq V^*(A) - V^*(A_{-b,-s}).$$

Using the relationships between W^* and V^* above, this also implies that for any $b \in \mathbb{B}$ and $s \in \mathbb{S}$,

$$W^* - W_{-b,.}^* + W^* - W_{.,-s}^* \geq W^* - W_{-b,-s}^*. \quad (14)$$

Thus, buyers and sellers in the original two sided allocation problems are complements and (CC) holds. The proposition below then follows.

Proposition 1 *Ex post efficient trade is impossible in any one-to-one allocation problem.*

Proposition 1 implies that if buyers and sellers have unit demand and unit supply or are exogenously restricted to a single partner such as in the model of Shapley and Shubik (1972), ex post efficient trade is impossible without running a deficit.

6 Matching Problems

As we saw in the previous section, connecting a one-to-one allocation problem to a related assignment game allows us to use existing results by Shapley (1962) to extend the impossibility results of Myerson and Satterthwaite (1983) to a general class of economic problems. This section shows that we can use a similar correspondence to establish impossibility results in many-to-many allocation problems.

We begin in Section 6.1 by defining **matching problems**. In Section 6.2 we use a generalization of Shapley (1962) to show that (CC) holds in any allocation problem that is a matching problem. This result provides sufficient conditions for the impossibility theorem to hold in many-to-many settings. In Section 7, we return to the primitives of the allocation problem and characterize matching problems.

6.1 Definition of a Matching Problem

Consider the assignment game defined by a payoff matrix A of dimension $(B+1)O \times O$. The rows of the payoff matrix represent **replicas** of the buyers and sellers while the columns of the matrix represent objects. To allow for the matrix to encode the assignment of any feasible package to a buyer, each buyer in the original allocation problem is represented in the matrix as O replicas. There are O additional rows to represent the sellers, resulting in a total of $(B+1)O$ rows. Each column of the matrix represents one of the O objects in the economy.

Let $\hat{\mathbb{B}}$ be the set of buyer replicas with typical element \hat{b} and cardinality $\hat{B} = B \cdot O$. Let $\hat{\mathbb{S}}$ be the set of seller replicas with typical element \hat{s} and cardinality $\hat{S} = O$. Then $\hat{\mathbb{R}} := \hat{\mathbb{B}} \cup \hat{\mathbb{S}}$ is the set of all replicas, with typical element r . The cardinality of this set, $(B+1)O$, is equal to the number of rows of the payoff matrix.

It is useful to also define $\hat{\mathbb{B}}_b \subseteq \hat{\mathbb{B}}$ to be the set of replicas of buyer b with cardinality O . Likewise we define $\hat{\mathbb{S}}_s \subseteq \hat{\mathbb{S}}$ to be the set of replicas of seller s with cardinality O_s . We also define the set of replicas $\hat{\mathbb{I}}$ to be the replicas corresponding to the buyers in set \mathbb{I} , i.e., $\hat{\mathbb{B}}_b \subseteq \hat{\mathbb{I}}$ if and only if $b \in \mathbb{I}$. The set of seller replicas $\hat{\mathbb{J}}$ corresponds to sellers in set \mathbb{J} in an analogous way. Finally, we denote by $\mathbb{O}_{\mathbb{J}} := \cup_{s \in \mathbb{J}} \mathbb{O}_s$ the set of objects in the potential set of a seller in set \mathbb{J} .

Definition 1 (Replica-Object Assignment Game) *A replica-object assignment game of a two-sided allocation problem is a $(B+1)O \times O$ matrix $A = (a_{r,o})_{r \in \hat{\mathbb{R}}, o \in \mathbb{O}}$ with nonnegative elements satisfying $a_{r,o} = 0$ for all (r,o) such that $r \in \hat{\mathbb{S}}_s$ and $o \in \mathbb{O}_{s'}$ with $s \neq s'$.*

In the original two-sided allocation problem, objects could be (i) produced by a seller and allocated to a buyer, (ii) produced by a seller and destroyed, or (iii) not produced. These allocations are expressed in the matching matrix in the following way. When an object $o_s \in \mathbb{O}_s$ is assigned to a replica of buyer b , this implies that the object is produced by seller s and traded to buyer b . When an object $o_s \in \mathbb{O}_s$ is not assigned to any replica, this implies the object is

produced by seller s and destroyed. Finally, when an object is assigned to a seller s , this implies that the object is not produced. In this way, the assignment game encodes an analogue of the two-sided allocation problem, where each seller is initially required to produce all objects in his potential set and objects are assigned to buyer replicas for consumption or seller replicas to reduce required production. For buyer replicas, the payoff matrix encodes the utility that the buyer receives from consuming an object. For seller replicas, the matrix encodes the cost savings from not having to produce an object. The restriction in Definition 1 that $a_{r,o} = 0$ if $r \in \hat{\mathbb{S}}_s$ and $o \in \mathbb{O}_{s'}$ with $s \neq s'$ embeds the constraint that sellers can only produce objects from their own potential set.

The maximum value created by assignment game A can be calculated by solving the following linear programming problem:

$$\begin{aligned}
V^*(A) &:= \max_L \sum_{r \in \hat{\mathbb{R}}} \sum_{o \in \mathbb{O}} l_{r,o} a_{r,o} \\
\text{subject to} & \quad \sum_{o \in \mathbb{O}} l_{r,o} \leq 1 \quad \text{for all } r \in \hat{\mathbb{R}} \\
& \quad \text{and } \sum_{r \in \hat{\mathbb{R}}} l_{r,o} \leq 1 \quad \text{for all } o \in \mathbb{O}.
\end{aligned} \tag{15}$$

It will be useful to define the assignment game in which a subset of replicas and a subset of objects have been removed. For any $\hat{\mathbb{T}} \subseteq \hat{\mathbb{R}}$, and $\mathbb{K} \subseteq \mathbb{O}$, let $A_{-\hat{\mathbb{T}},-\mathbb{K}}$ be defined from matrix A by removing all rows relating to replicas in $\hat{\mathbb{T}}$ and all columns relating to objects in \mathbb{K} .¹⁴

For any replica-object assignment game A and any allocation $x \in X$, a feasible matching L is said to be **isomorphic** to x if for any $o \in \mathbb{O}$,

$$\begin{aligned}
\sum_{r \in \hat{\mathbb{B}}_b} l_{r,o} &= \begin{cases} 1, & \text{if } o \in x_b \\ 0, & \text{otherwise} \end{cases} \quad \text{for all } b \in \mathbb{B} \quad \text{and} \\
\sum_{r \in \hat{\mathbb{S}}_s} l_{r,o} &= \begin{cases} 1, & \text{if } o \in \mathbb{O}_s \setminus x_s \\ 0, & \text{otherwise} \end{cases} \quad \text{for all } s \in \mathbb{S}.
\end{aligned}$$

When L is isomorphic to x , replicas of each buyer b are jointly assigned x_b and replicas of each seller s are jointly assigned $\mathbb{O}_s \setminus x_s$, which are the objects that s does not have to produce under allocation x . Let $\mathbb{L}(A, x)$ be the subset of feasible matchings of A that are isomorphic to x .

¹⁴Notice that $A_{-\hat{\mathbb{T}},-\mathbb{K}}$ will not necessarily satisfy Definition 1. In fact $A_{-\hat{\mathbb{T}},-\mathbb{K}}$ is a replica-object assignment game of a two-sided allocation problem if and only if for some $\mathbb{I} \in \mathbb{B}$ and $\mathbb{J} \subseteq \mathbb{S}$, $\hat{\mathbb{T}} = \hat{\mathbb{I}} \cup \hat{\mathbb{J}}$ and $\mathbb{K} = \mathbb{O}_{\mathbb{J}}$. That is, for each buyer either none or all of his replicas are removed and for each seller either none or all of his replicas and objects are removed. Observe however that the maximum value of any assignment game, whether or not it satisfies Definition 1, can be computed using (15).

Let $\mathbb{L}^*(A, x) \subseteq \mathbb{L}(A, x)$ be the set of matchings that are isomorphic to x and return the maximal value. We refer to these as the set of **best isomorphic matchings** of allocation x in assignment game A . Formally, for any $L \in \mathbb{L}(A, x)$, $L \in \mathbb{L}^*(A, x)$ if for all $L' \in \mathbb{L}(A, x)$,

$$\sum_{r \in \hat{\mathbb{R}}} \sum_{o \in \mathbb{O}} l_{r,o} a_{r,o} \geq \sum_{r \in \hat{\mathbb{R}}} \sum_{o \in \mathbb{O}} l'_{r,o} a_{r,o}. \quad (16)$$

Where there is no risk of confusion, we write L^x for a best isomorphic matching of x in A . If $\mathbb{L}^*(A, x)$ contains more than one element, then L^x can be arbitrarily chosen among them. It is also useful to define x^0 to be the allocation where nothing is produced ($x_b^0 = \emptyset$ for all b and $x_s^0 = \emptyset$ for all s). Let $L^{x^0} \in \mathbb{L}^*(A, x^0)$ be a best isomorphic matching of x^0 . Notice that L^{x^0} matches all replicas of each seller to an object in that seller's potential set but leaves all buyer replicas unmatched.

We can now formally define a matching problem:

Definition 2 (Matching Problem) *A two-sided allocation problem is a matching problem if for all (\mathbf{v}, \mathbf{c}) there exists a replica-object assignment game A satisfying Definition 1 such that for any feasible allocation $x \in X$ and any best isomorphic matching L^x ,*

$$\begin{aligned} \bar{u}_b(x_b, \mathbf{v}_b) &= \sum_{r \in \hat{\mathbb{B}}_b} \sum_{o \in \mathbb{O}} l_{r,o}^x a_{r,o} \quad \text{for all } b \in \mathbb{B} \quad \text{and} \\ \underline{k}_s(x_s, \mathbf{c}_s) &= \sum_{r \in \hat{\mathbb{S}}_s} \sum_{o \in \mathbb{O}} [l_{r,o}^{x^0} a_{r,o} - l_{r,o}^x a_{r,o}] \quad \text{for all } s \in \mathbb{S}. \end{aligned}$$

This definition of matching problems is natural insofar as all the information needed to describe an allocation, the welfare consequences of an allocation, and the resulting trade network is encoded in a one-to-one match between agent replicas and objects in a bipartite graph. The utility that a buyer receives from consuming a package equals the sum of the payoffs generated from assigning that package to replicas of that buyer. Likewise, for any seller, there is an exact correspondence between the cost for producing a package and the payoff that is generated from objects *not* assigned to replicas of that seller.

In the assignment game with all agents present, a buyer's utility and a seller's cost function are independent of the package assigned to other agents. In order for Definition 2 to hold, it must also be the case that in the best isomorphic matching of x , the payoffs generated by an assignment of a package to replicas of a buyer or a seller must be independent of the assignments made to the replicas of the other agents. We show that this independence allows

us to maintain a correspondence between the two-sided allocation problem and an assignment game even when subsets of buyers and sellers are removed.

Consider a two-sided allocation problem obtained by removing a subset of buyers $\mathbb{I} \subseteq \mathbb{B}$, a subset of sellers $\mathbb{J} \subseteq \mathbb{S}$, and the objects that correspond to these sellers $\mathbb{O}_{\mathbb{J}}$. Let $x = ((x_b)_{b \in \mathbb{B} \setminus \mathbb{I}}, (x_s)_{s \in \mathbb{S} \setminus \mathbb{J}})$ be an allocation in this smaller two-sided allocation problem. Let $A_{-\hat{\mathbb{I}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}}$ be a submatrix of A with the rows corresponding to replicas $\hat{\mathbb{I}} \cup \hat{\mathbb{J}}$ and objects $\mathbb{O}_{\mathbb{J}}$ removed. Note that $A_{-\hat{\mathbb{I}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}}$ continues to satisfy Definitions 1 and 2 for any allocation x . This implies that the smaller two-sided allocation problem is a matching problem and has $A_{-\hat{\mathbb{I}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}}$ as a corresponding assignment game. Let $L^x \in \mathbb{L}^*(A_{-\hat{\mathbb{I}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}}, x)$ be a best isomorphic matching of x in $A_{-\hat{\mathbb{I}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}}$. Then, the following lemma implies that social welfare in this best isomorphic matching is equal to the maximal welfare generated from the corresponding assignment game up to a constant.

Lemma 1 *Consider a matching problem with types (\mathbf{v}, \mathbf{c}) and corresponding replica-object assignment game A . For any subsets of agents $\mathbb{I} \subseteq \mathbb{B}$ and $\mathbb{J} \subseteq \mathbb{S}$ and any allocation $x \in X$ of the smaller matching problem obtained by removing these agents and their objects, let $L^x \in \mathbb{L}^*(A_{-\hat{\mathbb{I}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}}, x)$ be a best isomorphic matching of x in the replica-object assignment game $A_{-\hat{\mathbb{I}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}}$. Then,*

$$W_{-\mathbb{I}, -\mathbb{J}}(x, \mathbf{v}, \mathbf{c}) = \sum_{r \in (\hat{\mathbb{B}} \setminus \hat{\mathbb{I}}) \cup (\hat{\mathbb{S}} \setminus \hat{\mathbb{J}})} \sum_{o \in \mathbb{O}} l_{r,o}^x a_{r,o} - V^*(A_{-\hat{\mathbb{B}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}})$$

and

$$W_{-\mathbb{I}, -\mathbb{J}}^* = V^*(A_{-\hat{\mathbb{I}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}}) - V^*(A_{-\hat{\mathbb{B}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}}).$$

Lemma 1 shows that there is a correspondence between W^* and $V^*(A)$ in all matching problems even when subsets of buyers and sellers are removed. The term $V^*(A_{-\hat{\mathbb{B}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}})$ changes only with the number of included sellers and not with the match because of the restriction that all elements of A must be nonnegative. The term normalizes the output of the assignment game so that if all objects are optimally assigned to the sellers, the value is zero.

6.2 Impossibility in Matching Problems

We now show that ex post efficient trade is impossible in all matching problems. We begin by using Lemma 1 to derive a sufficient condition for (CC) in matching problems. We then show that this sufficient condition holds by extending the results of Shapley (1962).

Recall that in an efficient allocation, the trade network can be separated into components. In all matching problems, we can define $\hat{\mathbb{B}}(G) := \cup_{b \in G} \hat{\mathbb{B}}_b$ to be the set of replicas of buyers in component G , $\hat{\mathbb{S}}(G) := \cup_{s \in G} \hat{\mathbb{S}}_s$ to be the set of replicas of sellers in component G , and $\mathbb{O}(G) := \cup_{s \in \mathbb{S}(G)} \mathbb{O}_s$ to be the set of objects traded in component G . It is convenient to define $\hat{\mathbb{R}}(G) := \hat{\mathbb{B}}(G) \cup \hat{\mathbb{S}}(G)$ to be the set of replicas of agents in component G .

For any component G in an efficient allocation, Lemma 1 implies that

$$\sum_{b \in \mathbb{B}(G)} [W^* - W_{-b,.}^*] = \sum_{b \in \mathbb{B}(G)} [V^*(A) - V^*(A_{-\hat{\mathbb{B}}_b,.})],$$

and¹⁵

$$\sum_{s \in \mathbb{S}(G)} [W^* - W_{.,-s}^*] = \left[\sum_{s \in \mathbb{S}(G)} [V^*(A) - V^*(A_{-\hat{\mathbb{S}}_s, -\mathbb{O}_s})] \right] - \left[V^*(A_{-\hat{\mathbb{B}},.}) - V^*(A_{-\hat{\mathbb{B}} \cup \hat{\mathbb{S}}(G), -\mathbb{O}(G)}) \right].$$

It also implies that

$$W^* - W_{-\mathbb{B}(G), -\mathbb{S}(G)}^* = V^*(A) - V^*(A_{-\hat{\mathbb{R}}(G), -\mathbb{O}(G)}) - \left[V^*(A_{-\hat{\mathbb{B}},.}) - V^*(A_{-\hat{\mathbb{B}} \cup \hat{\mathbb{S}}(G), -\mathbb{O}(G)}) \right].$$

Because the final terms in these two equations are the same, it follows that (CC) holds in a matching problem if and only if

$$\sum_{b \in \hat{\mathbb{B}}(G)} [V^*(A) - V^*(A_{-\hat{\mathbb{B}}_b,.})] + \sum_{s \in \hat{\mathbb{S}}(G)} [V^*(A) - V^*(A_{-\hat{\mathbb{S}}_s, -\mathbb{O}_{\hat{\mathbb{S}}_s})}] \geq V^*(A) - V^*(A_{-\hat{\mathbb{R}}(G), -\mathbb{O}(G)}). \quad (17)$$

We now show that (17) holds for all assignment games by generalizing the substitutes condition of Shapley (1962). For an assignment game between replicas and objects, we say that **replicas are set substitutes** if for any two disjoint subsets of replicas $\hat{\mathbb{T}} \subseteq \hat{\mathbb{R}}$ and $\hat{\mathbb{T}}' \subseteq \hat{\mathbb{R}}$ with $\hat{\mathbb{T}} \cap \hat{\mathbb{T}}' = \emptyset$,

$$V^*(A) - V^*(A_{-\hat{\mathbb{T}},.}) + V^*(A) - V^*(A_{-\hat{\mathbb{T}}',.}) \leq V^*(A) - V^*(A_{-\hat{\mathbb{T}} \cup \hat{\mathbb{T}}',.}). \quad (18)$$

We say that **any number of replicas are substitutes** if for any subset of replicas $\hat{\mathbb{T}} \subseteq \hat{\mathbb{R}}$,

$$\sum_{r \in \hat{\mathbb{T}}} [V^*(A) - V^*(A_{-r,.})] \leq V^*(A) - V^*(A_{-\hat{\mathbb{T}},.}). \quad (19)$$

Likewise, **objects are set substitutes** if for any two disjoint subsets of objects $\mathbb{K} \subseteq \mathbb{O}$ and $\mathbb{K}' \subseteq \mathbb{O}$ with $\mathbb{K} \cap \mathbb{K}' = \emptyset$,

$$V^*(A) - V^*(A_{.,-\mathbb{K}}) + V^*(A) - V^*(A_{.,-\mathbb{K}'}) \leq V^*(A) - V^*(A_{.,-\mathbb{K} \cup \mathbb{K}'}), \quad (20)$$

¹⁵Note that $V^*(A_{-\hat{\mathbb{B}},.}) - V^*(A_{-\hat{\mathbb{B}} \cup \hat{\mathbb{S}}_s, -\mathbb{O}_s})$ is equivalent to the value of optimally assigning all objects in \mathbb{O}_s to replicas in $\hat{\mathbb{S}}_s$. Adding this value over all sellers in $\mathbb{S}(G)$ gives the same value as $V^*(A_{-\hat{\mathbb{B}},.}) - V^*(A_{-\hat{\mathbb{B}} \cup \hat{\mathbb{S}}(G), -\mathbb{O}(G)})$.

and **any number of objects are substitutes** if for any subset of objects $\mathbb{K} \subseteq \mathbb{O}$,

$$\sum_{o \in \mathbb{K}} [V^*(A) - V^*(A, -o)] \leq V^*(A) - V^*(A, -\mathbb{K}). \quad (21)$$

In addition, **replicas and objects are set complements** if for any $\hat{\mathbb{T}} \subseteq \hat{\mathbb{R}}$ and $\mathbb{K} \subseteq \mathbb{O}$,

$$V^*(A) - V^*(A_{-\hat{\mathbb{T}}}) + V^*(A) - V^*(A, -\mathbb{K}) \geq V^*(A) - V^*(A_{-\hat{\mathbb{T}}, -\mathbb{K}}). \quad (22)$$

Shapley (1962) shows that in an assignment game, any two replicas or any two objects are substitutes for each other. The following lemma generalizes this result, along with a notion of complements, to sets:

Lemma 2 *For any assignment game between replicas and objects, (i) replicas are set substitutes, (ii) any number of replicas are substitutes, (iii) objects are set substitutes, (iv) any number of objects are substitutes, and (v) replicas and objects are set complements.*

As was shown in Shapley (1962), buyer replicas and objects are complements in the assignment game, and thus the sum of their marginal products is greater than the sum for the pair. By Lemma 2, replicas of a given buyer are set substitutes and objects traded to this buyer are set substitutes. As shown in the appendix, these two conditions are enough to show that the marginal surplus lost from removing replicas corresponding to a buyer plus the marginal surplus lost from removing the objects optimally matched to that buyer exceeds the surplus lost from removing the corresponding trades. We can use this result along with a careful accounting of matches between objects and sellers to show that ex post efficient trade is impossible in all matching problems:

Theorem 3 *Ex post efficient trade is impossible in any matching problem.*

7 Matching Problems: A Characterization

We now return to the primitives of the two-sided allocation problem to answer the question: Under what conditions on buyers' utility functions and sellers' cost functions is a two-sided allocation problem a matching problem? In Section 7.1, we show that a necessary and sufficient condition for an allocation problem to be a matching problem is that each agent's payoff function can be derived as the solution to an assignment game. Such a representation exists if an agent can be decomposed into replicas and if a replica-object assignment game exists such

that the payoff to an agent from a package of objects corresponds to the optimal solution of the corresponding replica-object assignment game. In Section 7.2, we use this result to show that the impossibility result holds in a variety of economically relevant settings in which agents' payoff functions satisfy what we call **rank-dependent discounts**.

7.1 Decomposition

For a buyer $b \in \mathbb{B}$, let A^b be an $O \times O$ assignment game whose rows represent O replicas of buyer b and columns represent objects. Let $V^*(A^b)$ be the maximal value that can be generated from the assignment game. Assignment game A^b is a **buyer decomposition** of buyer b if for any package $x_b \in \mathbb{P}$

$$\bar{u}_b(x_b, \mathbf{v}_b) = V^*(A_{x_b}^b),$$

where $A_{x_b}^b$ is defined as the submatrix of A^b containing exclusively its columns related to objects in x_b . A buyer decomposition exists if it is possible to construct an assignment game whose maximal value coincides with the upper envelope of the buyer's utility function for any package of objects. Buyer b is **decomposable** if such a decomposition exists.¹⁶

The concept of decomposability can be defined in an almost analogous way for a seller. Let A^s be a $O_s \times O_s$ assignment game with maximal value $V^*(A^s)$. The assignment game A^s is a **seller decomposition** of seller $s \in \mathbb{S}$ if for any package $x_s \in \mathbb{P}_s$

$$\underline{\kappa}_s(x_s, \mathbf{c}_s) = V^*(A^s) - V^*(A_{\mathbb{O}_s \setminus x_s}^s),$$

where $A_{\mathbb{O}_s \setminus x_s}^s$ is defined as a submatrix of A^s containing exclusively the columns related to objects in $\mathbb{O}_s \setminus x_s$. A seller is decomposable if it is possible to construct an assignment game where the joint marginal payoff of the columns referring to objects in any package x_s is equal to the (lower envelope) cost of producing that package. Seller s is **decomposable** if such a decomposition exists.¹⁷

If all agents are decomposable, then all utility and cost functions can be represented by an assignment game. It is then possible to stack these matrices on top of each other to obtain a larger matrix that satisfies Definition 1. This implies that the two-sided allocation problem is a matching problem. If some agent is not decomposable, then such a matrix cannot be constructed.

¹⁶Hurwicz (1973) uses the term decomposable to refer to environments without externalities. In our setup, an agent's payoff function is decomposable if there are no externalities from one replica of the agent to any other.

¹⁷Notice that we have restricted the number of replicas to be equal to the maximum number of objects that an agent can consume or produce. This is without loss of generality, as shown in Lemma 3 in the appendix.

Theorem 4 *A two-sided allocation problem is a matching problem if and only if each buyer and seller is decomposable.*

7.2 Rank-Dependent Discounts Payoff Functions

Theorem 4 states decomposability as a necessary and sufficient condition for a two-sided allocation problem to be a matching problem, but leaves open the question under which conditions an agent is decomposable. In this section, we introduce a condition on payoff functions, called **rank-dependent discounts (RDD)**, and show that if an agent's payoff function exhibits RDD, then this agent is decomposable. The family of RDD payoff functions nests a wide range of payoff functions as special cases, including homogeneous objects, additive payoffs, unit demand and unit capacities, and an additive separable version of Ausubel's (2006) heterogeneous commodity model.

Let $\{\mathbb{O}_m\}_{m \in \mathbb{M}^b}$ be a partition of \mathbb{O} and denote the number of objects in block m by O_m . For any subset of objects $x_m \subseteq \mathbb{O}_m$, denote the object in this subset with the i th highest **stand-alone** utility, $\bar{u}_b(\{o\}, \mathbf{v}_b)$, as $o_{(i)}(x_m)$. That is:

$$\bar{u}_b(\{o_{(1)}(x_m)\}, \mathbf{v}_b) \geq \bar{u}_b(\{o_{(2)}(x_m)\}, \mathbf{v}_b) \geq \dots \geq \bar{u}_b(\{o_{(|x_m|)}(x_m)\}, \mathbf{v}_b). \quad (23)$$

Also let $\boldsymbol{\delta}_m = (\delta_{m,1}, \delta_{m,2}, \dots, \delta_{m,O_m})$ be a vector of rank-specific discount parameters of dimension O_m with $0 = \delta_{m,1} \leq \delta_{m,2} \leq \dots \leq \delta_{m,O_m}$.

For a block of objects \mathbb{O}_m , a buyer's payoff function exhibits RDD if for any $x_m \subseteq \mathbb{O}_m$,

$$\bar{u}_b(x_m, \mathbf{v}_b) = \sum_{i=1}^{|x_m|} \max\{\bar{u}_b(\{o_{(i)}(x_m)\}, \mathbf{v}_b) - \delta_{m,i}, 0\}. \quad (24)$$

Rank-dependent discounts allow individuals to have arbitrary stand-alone utilities but impose restrictions on the utility function as larger packages are assembled. As one example, recall the situation in Section 2, where David initially has preferences over objects A and B , with stand-alone utilities 9 and 5 and a utility for package AB of 12. These initial preferences exhibit RDD because for package AB , $u_b(\{o_{(1)}(AB)\}, \mathbf{v}_b) = 9$, $u_b(\{o_{(2)}(AB)\}, \mathbf{v}_b) = 5$, and $\delta_{m,2} = 2$. If there were a third object C with stand-alone utility of 7, then in order to exhibit RDD on ABC , David's utility for the package consisting of A and C would need to be 14 and his utility for the package consisting of B and C would need to be 10.

Definition 3 *A buyer's utility function exhibits RDD if there exists a partition $\{\mathbb{O}_m\}_{m \in \mathbb{M}^b}$*

and a collection of discount vectors $\boldsymbol{\delta} = \{\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_{M^b}\}$ such that for any $x \in \mathbb{P}$,

$$\bar{u}_b(x, \mathbf{v}_b) = \sum_{m \in \mathbb{M}^b} \sum_{i=1}^{|x_m|} \max\{\bar{u}_b(\{o_{(i)}(x_m)\}, \mathbf{v}_b) - \delta_{m,i}, 0\}, \quad (25)$$

where $x_m := x \cap \mathbb{O}_m$.

RDD requires that a buyer's utility function be additive across blocks and that, within each block, the value of a package be equal to the sum of the stand-alone utilities minus a discount that only depends on the number of units the package contains. These requirements are met in the homogeneous objects case with decreasing marginal utility since the stand-alone cost of all objects is the same and the addition of an extra unit has a fixed discount. The requirements is also satisfied in the case of additive utility because each object constitutes a separate block.

Analogous to the buyers, let $\{\mathbb{O}_{m,s}\}_{m \in M^s}$ be a partition of objects \mathbb{O}_s and denote the number of objects in block m by $O_{m,s}$. For any subset of objects $x_m \subseteq \mathbb{O}_{m,s}$, denote the object in this subset with the i th lowest stand-alone cost, $\underline{\kappa}_s(\{o\}, \mathbf{c}_s)$, as $o_{(i)}(x_m)$. Also let $\boldsymbol{\delta}_m = (\delta_{m,1}, \delta_{m,2}, \dots, \delta_{m,O_{m,s}})$ be a vector of rank-specific discount parameters of dimension $O_{m,s}$ with $0 = \delta_{m,1} \leq \delta_{m,2} \leq \dots \leq \delta_{m,O_{m,s}}$.

Definition 4 A seller's cost function exhibits RDD if there exists a partition $\{\mathbb{O}_{m,s}\}_{m \in M^s}$ and vector of discounts $\boldsymbol{\delta} = \{\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_{M^s}\}$ such that for any $x \in \mathbb{P}_s$,

$$\underline{\kappa}_s(x, \mathbf{c}_s) = \sum_{m \in \mathbb{M}^s} \sum_{i=1}^{|x_m|} \underline{\kappa}_s(\{o_{(i)}(x_m)\}, \mathbf{c}_s) + \delta_{m,i}, \quad (26)$$

where $x_m := x \cap \mathbb{O}_{m,s}$.

As with the buyer, a seller's payoff exhibits RDD if there exists a partition of the seller's potential set where costs are additive across blocks and, within each block, the cost of producing a package is equal to the sum of the stand-alone utilities plus a discount that only depends on the number of units the package contains.¹⁸ Such preferences may represent a situation where a seller has M^s production lines or factories and each production facility produces homogeneous objects.

The family of RDD payoff functions nests a wide range of payoff functions as special cases. As mentioned earlier, if all stand-alone utilities (costs) are the same, then the buyer (seller)

¹⁸Note that the seller's problem is the same as the buyers problem if one views the seller as initially owning his potential set and selling the ones he likes least first.

considers the objects to be homogeneous. If the discount parameters are very large then buyers and sellers are unit traders in the sense that they efficiently consume, respectively produce, packages containing at most one unit.¹⁹

The assignment game of Shapley and Shubik (1972) is a special case of this model because buyers are unit traders and the potential set of each seller contains exactly one object ($O_s = 1$ for all s). In contrast, if the discount parameters are zero, then the agent has additive payoffs. Ausubel (2006) studies a model in which the set of objects is partitioned into commodities, with each available in multiple identical units. The version of that model in which payoffs are additively separable across commodities ensures that all payoff functions exhibit RDD.

The following proposition shows that the general nature of RDD payoff functions allows us to classify various well-known models as matching problems.

Proposition 2 *An agent is decomposable if his payoff function exhibits RDD.*

Corollary 1 *Any two-sided allocation problem in which all agents' payoff functions exhibit RDD is a matching problem.*

An immediate implication of Corollary 1 is that ex post efficient trade is impossible in any two-sided allocation problem in which all payoff functions exhibit RDD. Because decomposability is determined at the individual level, Corollary 1 allows us to derive the impossibility theorem for setups in which agents have different payoff functions as long as all of them exhibit RDD. Thus, our results can be applied to settings in which buyers and sellers have very different characteristics.

7.3 Decomposability and Substitutes Preferences

In all the applications introduced above, objects are always substitutes for all buyers and all sellers. This is not a coincidence as the following proposition shows: An agent can only be decomposable if he perceives objects are as substitutes.

For a buyer b consuming a package $x \in \mathbb{P}$, two disjoint subsets of that package $y, z \subseteq x$ with $y \cap z = \emptyset$ are **substitutes for each other** if

$$\bar{u}_b(x, \mathbf{v}_b) - \bar{u}_b(x \setminus y, \mathbf{v}_b) \leq \bar{u}_b(x \setminus z, \mathbf{v}_b) - \bar{u}_b(x \setminus (y \cup z), \mathbf{v}_b),$$

¹⁹One-to-one allocation problems studied in Section 5 can be mapped into a matching problem with RDD payoffs by first determining the package that maximizes the joint surplus of each pair and then redefining cost and utility functions over these efficient packages. Agents in this transformed problem are unit traders.

that is, if the marginal utility of y weakly increases when z is removed. For a seller s producing a package $x \in \mathbb{P}_s$, two disjoint subsets of that package $y, z \subseteq x$ with $y \cap z = \emptyset$ are substitutes for each other if the marginal cost of y weakly decreases when z is removed:

$$\underline{\kappa}_s(x, \mathbf{c}_s) - \underline{\kappa}_s(x \setminus y, \mathbf{c}_s) \geq \underline{\kappa}_s(x \setminus z, \mathbf{c}_s) - \underline{\kappa}_s(x \setminus (y \cup z), \mathbf{c}_s).$$

Proposition 3 *An agent is decomposable only if any two disjoint subsets of any package are substitutes for one another.*

The intuition is clear. From Shapley (1962), we know that any two objects in a replica-object assignment game are substitutes for each other, which, by Lemma 2, extends to sets of objects. Because an assignment game does not allow any complementarity between objects, the payoff function in a two-sided allocation problem also cannot exhibit complementarity if it is to be decomposable.

8 Conclusion

We establish the impossibility of ex post efficient trade for general environments with buyers and sellers whose types are multi-dimensional. We develop a new component complements condition that is sufficient for all mechanisms that are dominant strategy incentive compatible and ex post individually rational to never run a surplus and least efficient buyer and seller types exist. When the condition holds strictly, all efficient incentive compatible mechanisms run a deficit.

For assignment games, which are two-sided allocation problems with one-to-one matching, a result due to Shapley (1962) establishes that buyers and sellers are complements. As this is sufficient for the component complements condition to hold, it implies the impossibility result for all one-to-one allocation problems. We generalize the impossibility result to setups in which trades are not necessarily one-to-one by first answering a question that is of independent interest: When is a two-sided allocation problem a matching problem? We show that a two-sided allocation problem is a matching problem – in the sense that the efficient allocation corresponds an optimal matching in an assignment game – if and only if each agent’s payoff function can be decomposed. Just like in Hurwicz (1973), decomposability means that there are no externalities, with the difference to Hurwicz’s setup being that here there are no externalities between replicas of the same agent. The impossibility result then generalizes to all two-sided allocation problems that are matching problems, that is, to all setups in which agents are decomposable.

We also introduce a new family of utility and cost functions, called rank-dependent discounts (RDD) payoff functions and show that RDD is sufficient for decomposability. RDD utility and cost functions nest a wide range of payoff functions as special cases, including homogeneous objects, additive payoffs, unit demands and supplies, and an additively separable version of the heterogeneous commodities model of Ausubel (2006).

Our research opens up a number of avenues for future study. While we have partially classified the set of many-to-many environments that can be mapped into matching problems, it may be useful to explore alternative conditions on preferences to study further the correspondence between many-to-many allocation problems and matching problems. RDD and decomposability are likely to prove useful in a variety of other contexts and applications. Decomposability makes the derivation of the efficient allocation computationally tractable. Because it is a condition on individual agents' payoffs, it can be verified without having to account for the interaction between different agents whose payoff functions may vary considerably. Lastly, decomposability and RDD may also be a necessary condition for efficiency to be achievable via clock auctions, which have a number of advantages over direct mechanisms.

A Appendix: Proofs

Proof of Lemma 1: By Definition 2,

$$\begin{aligned}\bar{u}_b(x_b, \mathbf{v}_b) &= \sum_{r \in \hat{\mathbb{B}}_b} \sum_{o \in \mathbb{O}} l_{r,o}^x a_{r,o} \quad \text{for all } b \in \mathbb{B} \setminus \mathbb{I}, \text{ and} \\ \underline{\kappa}_s(x_s, \mathbf{c}_s) &= \sum_{r \in \hat{\mathbb{S}}_s} \sum_{o \in \mathbb{O}} [l_{r,o}^{x^0} a_{r,o} - l_{r,o}^x a_{r,o}] \quad \text{for all } s \in \mathbb{S} \setminus \mathbb{J},\end{aligned}$$

where x^0 is the allocation that leaves all remaining agents with an empty package ($x_b^0 = \emptyset$ for all $b \in \mathbb{B} \setminus \mathbb{I}$ and $x_s^0 = \emptyset$ for all $s \in \mathbb{S} \setminus \mathbb{J}$) and L^{x^0} is a best isomorphic matching of that allocation in $A_{-\hat{\mathbb{I}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}}$. From (4) recall that social welfare $W_{-\mathbb{I}, -\mathbb{J}}(x, \mathbf{v}, \mathbf{c})$ is

$$W_{-\mathbb{I}, -\mathbb{J}}(x, \mathbf{v}, \mathbf{c}) = \sum_{b \in \mathbb{B} \setminus \mathbb{I}} \bar{u}_b(x_b, \mathbf{v}_b) - \sum_{s \in \mathbb{S} \setminus \mathbb{J}} \underline{\kappa}_s(x_s, \mathbf{c}_s).$$

Combining these two results yields

$$\begin{aligned}W_{-\mathbb{I}, -\mathbb{J}}(x, \mathbf{v}, \mathbf{c}) &= \sum_{b \in \mathbb{B} \setminus \mathbb{I}} \sum_{r \in \hat{\mathbb{B}}_b} \sum_{o \in \mathbb{O}} l_{r,o}^x a_{r,o} - \sum_{s \in \mathbb{S} \setminus \mathbb{J}} \sum_{r \in \hat{\mathbb{S}}_s} \sum_{o \in \mathbb{O}} [l_{r,o}^{x^0} a_{r,o} - l_{r,o}^x a_{r,o}] \\ &= \sum_{r \in (\hat{\mathbb{B}} \setminus \hat{\mathbb{I}}) \cup (\hat{\mathbb{S}} \setminus \hat{\mathbb{J}})} \sum_{o \in \mathbb{O}} l_{r,o}^x a_{r,o} - \sum_{r \in \hat{\mathbb{S}} \setminus \hat{\mathbb{J}}} \sum_{o \in \mathbb{O}} l_{r,o}^{x^0} a_{r,o}.\end{aligned}$$

Recalling that L^0 optimally matches the replicas of each seller s with the objects in \mathbb{O}_s ,

$$\sum_{r \in \hat{\mathbb{S}} \setminus \hat{\mathbb{J}}} \sum_{o \in \mathbb{O}} l_{r,o}^{x^0} a_{r,o} = V^*(A_{-\hat{\mathbb{B}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}}) \quad \text{for any } \mathbb{J} \subseteq \mathbb{S}.$$

Combining the last two definitions completes the proof of the first part of the statement,

$$W_{-\mathbb{I}, -\mathbb{J}}(x, \mathbf{v}, \mathbf{c}) = \sum_{r \in (\hat{\mathbb{B}} \setminus \hat{\mathbb{I}}) \cup (\hat{\mathbb{S}} \setminus \hat{\mathbb{J}})} \sum_{o \in \mathbb{O}} l_{r,o}^x a_{r,o} - V^*(A_{-\hat{\mathbb{B}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}}) \quad \square \quad (27)$$

To complete the proof, it remains to show that L^{x^*} is an optimal matching of $A_{-\hat{\mathbb{I}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}}$. To see this, consider any feasible matching of $A_{-\hat{\mathbb{I}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}}$. Because it assigns each object at most once, it is an isomorphic matching of some allocation. The output it produces is therefore at most the output produced by a best isomorphic matching of that allocation. By (27) and because x^* is efficient, this cannot exceed the output created by L^{x^*} , which is equal to $W_{-\mathbb{I}, -\mathbb{J}}^* + V^*(A_{-\hat{\mathbb{B}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}})$. The latter is therefore an optimal matching of $A_{-\hat{\mathbb{I}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}}$. Consequently, $\sum_{r \in (\hat{\mathbb{B}} \setminus \hat{\mathbb{I}}) \cup (\hat{\mathbb{S}} \setminus \hat{\mathbb{J}})} \sum_{o \in \mathbb{O}} l_{r,o}^{x^*} a_{r,o} = V^*(A_{-\hat{\mathbb{I}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}})$ and

$$W_{-\mathbb{I}, -\mathbb{J}}^* = V^*(A_{-\hat{\mathbb{I}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}}) - V^*(A_{-\hat{\mathbb{B}} \cup \hat{\mathbb{J}}, -\mathbb{O}_{\mathbb{J}}}). \quad \blacksquare$$

Proof of Lemma 2: We begin by proving part (ii), showing that (19) holds. Observe first that (19) holds with an equality if $\hat{T} \leq 1$, so we assume that $\hat{T} \geq 2$. Label the elements of $\hat{\mathbb{T}}$ such that $\hat{\mathbb{T}} = \{r_1, \dots, r_{\hat{T}}\}$ and define $\hat{\mathbb{T}}_n := \{r_1, \dots, r_n\}$ for any $n \in \{1, \dots, \hat{T}\}$ as well as $\hat{\mathbb{T}}_0 := \emptyset$. By Theorem 2, for any $m \in \{2, \dots, \hat{T}\}$ and $n \in \{1, \dots, m-1\}$, r_m and r_n are substitutes for each other in the assignment game defined by matrix $A_{-\hat{\mathbb{T}}_{n-1}, \cdot}$. That is, for any $m \in \{2, \dots, \hat{T}\}$ and $n \in \{1, \dots, m-1\}$,

$$V^*(A_{-\hat{\mathbb{T}}_{n-1}, \cdot}) - V^*(A_{-\hat{\mathbb{T}}_{n-1} \cup \{r_m\}, \cdot}) + V^*(A_{-\hat{\mathbb{T}}_{n-1}, \cdot}) - V^*(A_{-\hat{\mathbb{T}}_n, \cdot}) \leq V^*(A_{-\hat{\mathbb{T}}_{n-1}, \cdot}) - V^*(A_{-\hat{\mathbb{T}}_n \cup \{r_m\}, \cdot}),$$

which can be rearranged as

$$V^*(A_{-\hat{\mathbb{T}}_{n-1}, \cdot}) - V^*(A_{-\hat{\mathbb{T}}_{n-1} \cup \{r_m\}, \cdot}) \leq V^*(A_{-\hat{\mathbb{T}}_n, \cdot}) - V^*(A_{-\hat{\mathbb{T}}_n \cup \{r_m\}, \cdot}). \quad (28)$$

Evaluating (28) at $n = 1, \dots, m-1$ gives us a series of $m-1$ inequalities of the form

$$\begin{aligned} V^*(A) - V^*(A_{-r_m, \cdot}) &\leq V^*(A_{-r_1, \cdot}) - V^*(A_{-\{r_1, r_m\}, \cdot}) \\ V^*(A_{-r_1, \cdot}) - V^*(A_{-\{r_1, r_m\}, \cdot}) &\leq V^*(A_{-\{r_1, r_2\}, \cdot}) - V^*(A_{-\{r_1, r_2, r_m\}, \cdot}) \\ &\dots \\ V^*(A_{-\hat{\mathbb{T}}_{m-2}, \cdot}) - V^*(A_{-\hat{\mathbb{T}}_{m-2} \cup \{r_m\}, \cdot}) &\leq V^*(A_{-\hat{\mathbb{T}}_{m-1}, \cdot}) - V^*(A_{-\hat{\mathbb{T}}_m, \cdot}), \end{aligned}$$

which implies

$$V^*(A) - V^*(A_{-r_m, \cdot}) \leq V^*(A_{-\hat{\mathbb{T}}_{m-1}, \cdot}) - V^*(A_{-\hat{\mathbb{T}}_m, \cdot}). \quad (29)$$

Because (28) holds for any $m \in \{2, \dots, \hat{T}\}$, so does (29). When $m = 1$, $\hat{\mathbb{T}}_{m-1} = \hat{\mathbb{T}}_0 \equiv \emptyset$, in which case (29) is satisfied with an equality. Therefore (29) implies

$$\begin{aligned} \sum_{m=1}^{\hat{T}} [V^*(A) - V^*(A_{-r_m, \cdot})] &\leq \sum_{m=1}^{\hat{T}} [V^*(A_{-\hat{\mathbb{T}}_{m-1}, \cdot}) - V^*(A_{-\hat{\mathbb{T}}_m, \cdot})] \\ &= V^*(A_{-\hat{\mathbb{T}}_0, \cdot}) - V^*(A_{-\hat{\mathbb{T}}_{\hat{T}}, \cdot}) \\ &= V^*(A) - V^*(A_{-\hat{\mathbb{T}}, \cdot}), \end{aligned}$$

where the first equality expands the sum and cancels terms and the final equality uses $\hat{\mathbb{T}}_0 \equiv \emptyset$ and $\hat{\mathbb{T}}_{\hat{T}} \equiv \hat{\mathbb{T}}$. This completes the proof that (19) holds and so completes the proof of part (ii).

□

We now turn to the proof of part (i), showing that (18) holds. Similar to the case above, (18) holds with an equality if either $\hat{\mathbb{T}} = \emptyset$ or $\hat{\mathbb{T}}' = \emptyset$, so we focus on the case in which both

subsets contain at least one element. As before, let $\hat{\mathbb{T}}_n := \{r_1, \dots, r_n\}$ for any $n \in \{1, \dots, \hat{T}\}$ and, analogously, let $\hat{\mathbb{T}}'_m := \{r'_1, \dots, r'_m\}$ for any $m \in \{1, \dots, \hat{T}'\}$. Define $\hat{\mathbb{T}}_0 := \emptyset$ and $\hat{\mathbb{T}}'_0 := \emptyset$.

By Theorem 2, for any $n \in \{1, \dots, \hat{T}\}$ and $m \in \{1, \dots, \hat{T}'\}$, r_n and r'_m are substitutes for each other in the assignment game defined by matrix $A_{-\hat{\mathbb{T}}_{n-1} \cup \hat{\mathbb{T}}'_{m-1}, \cdot}$. That is, for any $n \in \{1, \dots, \hat{T}\}$ and $m \in \{1, \dots, \hat{T}'\}$,

$$\begin{aligned} V^*(A_{-\hat{\mathbb{T}}_{n-1} \cup \hat{\mathbb{T}}'_{m-1}, \cdot}) - V^*(A_{-\hat{\mathbb{T}}_n \cup \hat{\mathbb{T}}'_{m-1}, \cdot}) + V^*(A_{-\hat{\mathbb{T}}_{n-1} \cup \hat{\mathbb{T}}'_m, \cdot}) - V^*(A_{-\hat{\mathbb{T}}_{n-1} \cup \hat{\mathbb{T}}'_m, \cdot}) \\ \leq V^*(A_{-\hat{\mathbb{T}}_{n-1} \cup \hat{\mathbb{T}}'_{m-1}, \cdot}) - V^*(A_{-\hat{\mathbb{T}}_n \cup \hat{\mathbb{T}}'_m, \cdot}). \end{aligned}$$

By rearranging the above inequality we obtain

$$V^*(A_{-\hat{\mathbb{T}}_{n-1} \cup \hat{\mathbb{T}}'_m, \cdot}) - V^*(A_{-\hat{\mathbb{T}}_n \cup \hat{\mathbb{T}}'_{m-1}, \cdot}) \leq V^*(A_{-\hat{\mathbb{T}}_{n-1} \cup \hat{\mathbb{T}}'_m, \cdot}) - V^*(A_{-\hat{\mathbb{T}}_n \cup \hat{\mathbb{T}}'_m, \cdot}). \quad (30)$$

By an analogous argument to the one developed in the proof of part (ii), keeping n fixed and invoking (30) for each possible value of m between 1 and \hat{T}' yields a series of \hat{T}' inequalities such that the right side of each is equal to the left side of the next. The left side of the first one must therefore be weakly smaller than the right side of the last one, that is

$$V^*(A_{-\hat{\mathbb{T}}_{n-1}, \cdot}) - V^*(A_{-\hat{\mathbb{T}}_n, \cdot}) \leq V^*(A_{-\hat{\mathbb{T}}_{n-1} \cup \hat{\mathbb{T}}'_1, \cdot}) - V^*(A_{-\hat{\mathbb{T}}_n \cup \hat{\mathbb{T}}'_1, \cdot}),$$

which is equivalent to

$$V^*(A_{-\hat{\mathbb{T}}_{n-1}, \cdot}) - V^*(A_{-\hat{\mathbb{T}}_{n-1} \cup \hat{\mathbb{T}}'_1, \cdot}) \leq V^*(A_{-\hat{\mathbb{T}}_n, \cdot}) - V^*(A_{-\hat{\mathbb{T}}_n \cup \hat{\mathbb{T}}'_1, \cdot}).$$

Because this inequality holds for any $n \in \{1, \dots, \hat{T}\}$, analogous reasoning to above yields

$$V^*(A) - V^*(A_{-\hat{\mathbb{T}}'_1, \cdot}) \leq V^*(A_{-\hat{\mathbb{T}}_1, \cdot}) - V^*(A_{-\hat{\mathbb{T}}_1 \cup \hat{\mathbb{T}}'_1, \cdot}),$$

which is equivalent to (18) and completes the proof of part (i). \square

By the symmetry of assignment games, the proofs of parts (iii) and (iv) are analogous. \square

We conclude with the proof of part (v), showing that (22) holds. Similarly to the proofs of parts (i) and (ii), (22) holds with an equality if either $\hat{\mathbb{T}} = \emptyset$ or $\mathbb{K} = \emptyset$, hence we focus on the case in which both subsets contain at least one element. As before, let $\hat{\mathbb{T}}_n := \{r_1, \dots, r_n\}$ for any $n \in \{1, \dots, \hat{T}\}$ and, analogously, let $\mathbb{K}_m := \{o_1, \dots, o_m\}$ for any $m \in \{1, \dots, K\}$. Define $\hat{\mathbb{T}}_0 := \emptyset$ and $\mathbb{K}_0 := \emptyset$.

By Theorem 2, for any $n \in \{1, \dots, \hat{T}\}$ and $m \in \{1, \dots, K\}$, r_n and o_m are complements to each other in the assignment game defined by matrix $A_{-\hat{\mathbb{T}}_{n-1}, -\mathbb{K}_{m-1}}$. That is, for any $n \in \{1, \dots, \hat{T}\}$

and $m \in \{1, \dots, K\}$,

$$\begin{aligned} V^*(A_{-\hat{\mathbb{T}}_{n-1}, -\mathbb{K}_{m-1}}) - V^*(A_{-\hat{\mathbb{T}}_n, -\mathbb{K}_{m-1}}) + V^*(A_{-\hat{\mathbb{T}}_{n-1}, -\mathbb{K}_{m-1}}) - V^*(A_{-\hat{\mathbb{T}}_{n-1}, -\mathbb{K}_m}) \\ \leq V^*(A_{-\hat{\mathbb{T}}_{n-1}, -\mathbb{K}_{m-1}}) - V^*(A_{-\hat{\mathbb{T}}_n, -\mathbb{K}_m}). \end{aligned}$$

By rearranging the above inequality we obtain

$$V^*(A_{-\hat{\mathbb{T}}_{n-1}, -\mathbb{K}_{m-1}}) - V^*(A_{-\hat{\mathbb{T}}_n, -\mathbb{K}_{m-1}}) \leq V^*(A_{-\hat{\mathbb{T}}_{n-1}, -\mathbb{K}_m}) - V^*(A_{-\hat{\mathbb{T}}_n, -\mathbb{K}_m}). \quad (31)$$

The remainder of the proof of part (v) is analogous to the proof of part (ii) starting after (28).

■

Proof of Theorem 3: We have established in the main text that (17) is equivalent to (CC) and therefore a sufficient condition for the impossibility of ex post efficient trade. The remainder of the proof is therefore devoted to proving that (17) holds for any replica-object assignment game A .

Let L^* be an optimal matching of A such that

- (i) $\sum_{r \in \hat{\mathbb{R}}} l_{r,o}^* = 1$ for all $o \in \mathbb{O}$, and
- (ii) $l_{r,o}^* = 0$ for all pairs (r, o) such that $r \in \hat{\mathbb{S}}_s$ and $o \in \mathbb{O}_{s'}$ where $s, s' \in \mathbb{S}$ with $s \neq s'$

An optimal matching satisfying (i) and (ii) always exists: An optimal matching satisfying (i), which states no object is left unmatched, always exists because objects are on the short side and the elements of A are nonnegative. Moreover, by Definition 1, $a_{r,o} = 0$ if $r \in \hat{\mathbb{S}}_s$ and $o \in \mathbb{O}_{s'}$ with $s \neq s'$. Consequently, one can always find an optimal matching that only matches seller replicas with objects that belong to their seller's potential set, as required by (ii).

For all $o \in \mathbb{O}$, let $s(o) \in \mathbb{S}$ be such that $o \in \mathbb{O}_{s(o)}$ and let $r^*(o) \in \hat{\mathbb{B}} \cup \{s(o)\}$ be the replica that L^* matches with object o . Then $l_{r^*(o),o}^* = 1$ for all $o \in \mathbb{O}$ and $l_{r,o}^* = 0$ for all (r, o) with $r \neq r^*(o)$. It follows that

$$V^*(A) = \sum_{r \in \hat{\mathbb{R}}} \sum_{o \in \mathbb{O}} l_{r,o}^* a_{r,o} = \sum_{o \in \mathbb{O}} a_{r^*(o),o}.$$

Let $\tilde{\mathbb{O}}_s := \{o \in \mathbb{O}_s \mid r^*(o) \in \hat{\mathbb{B}}\}$ be the subset of objects in seller s 's potential set that are matched with a buyer replica. Then $\mathbb{O}_s \setminus \tilde{\mathbb{O}}_s$ represents the set of objects matched with replicas of s . Let $\tilde{\mathbb{O}}(G) := \cup_{s \in \mathbb{S}(G)} \tilde{\mathbb{O}}_s = \{o \in \mathbb{O}(G) \mid r^*(o) \in \hat{\mathbb{B}}(G)\}$ be the set of objects matched to a replica of a buyer in component G .

Theorem 2 implies that the marginal product of each replica and each object exceeds the marginal product generated on the link. Adding all links between buyer replicas and objects in a given component G , this implies that:

$$\sum_{o \in \tilde{\mathcal{O}}(G)} [V^*(A) - V^*(A_{-r^*(o),.}) + V^*(A) - V^*(A_{.,-o})] \geq \sum_{o \in \tilde{\mathcal{O}}(G)} [V^*(A) - V^*(A_{-r^*(o),-o})]. \quad (32)$$

Because o and $r^*(o)$ are optimally matched together, removing that pair does not affect the rest of the assignment game. Consequently, its marginal product is $V^*(A) - V^*(A_{-r^*(o),-o}) = a_{r^*(o),o}$, and the right side of (32) is equal to $\sum_{o \in \tilde{\mathcal{O}}(G)} a_{r^*(o),o}$.

Notice next that

$$\sum_{o \in \tilde{\mathcal{O}}(G)} [V^*(A) - V^*(A_{-r^*(o),.})] = \sum_{\hat{b} \in \hat{\mathbb{B}}(G)} [V^*(A) - V^*(A_{-\hat{b},.})]$$

because each object in $\tilde{\mathcal{O}}(G)$ is matched to a buyer-replica of component G and each one of the latter is either unmatched or matched to an object in $\tilde{\mathcal{O}}(G)$. By part (ii) of Lemma 2, for any $b \in \mathbb{B}(G)$ we have

$$V^*(A) - V^*(A_{-\hat{\mathbb{B}}_b,.}) \geq \sum_{\hat{b} \in \hat{\mathbb{B}}_b} [V^*(A) - V^*(A_{-\hat{b},.})],$$

while part (iv) of Lemma 2 implies for any $s \in \mathcal{S}(G)$

$$V^*(A) - V^*(A_{.,-\tilde{\mathcal{O}}_s}) \geq \sum_{\hat{o} \in \tilde{\mathcal{O}}_s} [V^*(A) - V^*(A_{.,-o})].$$

Adding up these terms over the buyers and sellers in component G yields

$$\begin{aligned} \sum_{b \in \mathbb{B}(G)} [V^*(A) - V^*(A_{-\hat{\mathbb{B}}_b,.})] &\geq \sum_{\hat{b} \in \hat{\mathbb{B}}(G)} [V^*(A) - V^*(A_{-\hat{b},.})] \\ \text{and } \sum_{s \in \mathcal{S}(G)} [V^*(A) - V^*(A_{.,-\tilde{\mathcal{O}}_s})] &\geq \sum_{o \in \tilde{\mathcal{O}}(G)} [V^*(A) - V^*(A_{.,-o})]. \end{aligned}$$

Using these results along with (32) yields

$$\sum_{b \in \mathbb{B}(G)} [V^*(A) - V^*(A_{-\hat{\mathbb{B}}_b,.})] + \sum_{s \in \mathcal{S}(G)} [V^*(A) - V^*(A_{.,-\tilde{\mathcal{O}}_s})] \geq \sum_{o \in \tilde{\mathcal{O}}(G)} a_{r^*(o),o}. \quad (33)$$

Removing replicas cannot increase the value created by an assignment game, that is, $V^*(A_{.,-\tilde{\mathcal{O}}_s}) \geq V^*(A_{-\hat{\mathcal{S}}_s,-\tilde{\mathcal{O}}_s})$. Combined with (33), this yields

$$\sum_{b \in \mathbb{B}(G)} [V^*(A) - V^*(A_{-\hat{\mathbb{B}}_b,.})] + \sum_{s \in \mathcal{S}(G)} [V^*(A) - V^*(A_{-\hat{\mathcal{S}}_s,-\tilde{\mathcal{O}}_s})] \geq \sum_{o \in \tilde{\mathcal{O}}(G)} a_{r^*(o),o}. \quad (34)$$

We next look at the marginal product of objects in $\mathbb{O}(G) \setminus \tilde{\mathbb{O}}(G)$. Each object in $\mathbb{O}_s \setminus \tilde{\mathbb{O}}_s$ is matched with a replica of s and each replica of s is either unmatched or matched with an object in $\mathbb{O}_s \setminus \tilde{\mathbb{O}}_s$, therefore for any $s \in \mathbb{S}$, removing $\hat{\mathbb{S}}_s$ and $\mathbb{O}_s \setminus \tilde{\mathbb{O}}_s$ implies that unmatched replicas and matched pairs are removed, hence for all $s \in \mathbb{S}$,

$$V^*(A) - V^*(A_{-\hat{\mathbb{S}}_s, -\mathbb{O}_s \setminus \tilde{\mathbb{O}}_s}) = \sum_{o \in \mathbb{O}_s \setminus \tilde{\mathbb{O}}_s} a_{r^*(o), o}.$$

Using this result in conjunction with (34) yields

$$\begin{aligned} & \sum_{b \in \mathbb{B}(G)} [V^*(A) - V^*(A_{-\hat{\mathbb{B}}_b, \cdot})] + \sum_{s \in \mathbb{S}(G)} \left[V^*(A) - V^*(A_{-\hat{\mathbb{S}}_s, -\tilde{\mathbb{O}}_s}) + V^*(A) - V^*(A_{-\hat{\mathbb{S}}_s, -\mathbb{O}_s \setminus \tilde{\mathbb{O}}_s}) \right] \\ & \geq \sum_{o \in \mathbb{O}(G)} a_{r^*(o), o}. \end{aligned}$$

We now invoke part (iii) of Lemma 2 to obtain, for all $s \in \mathbb{S}$,

$$V^*(A) - V^*(A_{-\hat{\mathbb{S}}_s, -\mathbb{O}_s}) \geq V^*(A) - V^*(A_{-\hat{\mathbb{S}}_s, -\tilde{\mathbb{O}}_s}) + V^*(A) - V^*(A_{-\hat{\mathbb{S}}_s, -\mathbb{O}_s \setminus \tilde{\mathbb{O}}_s}).$$

and thus

$$\sum_{b \in \mathbb{B}(G)} [V^*(A) - V^*(A_{-\hat{\mathbb{B}}_b, \cdot})] + \sum_{s \in \mathbb{S}(G)} [V^*(A) - V^*(A_{-\hat{\mathbb{S}}_s, -\mathbb{O}_s})] \geq \sum_{o \in \mathbb{O}(G)} a_{r^*(o), o}.$$

Observe finally that all objects in $\mathbb{O}(G)$ are matched to a replica in $\hat{\mathbb{R}}(G)$ and all replicas in $\hat{\mathbb{R}}(G)$ are either unmatched or matched to an object in $\mathbb{O}(G)$. Removing $\hat{\mathbb{R}}(G)$ and $\mathbb{O}(G)$ means removing matched pairs and unmatched replicas so

$$V^*(A) - V^*(A_{-\hat{\mathbb{R}}(G), -\mathbb{O}(G)}) = \sum_{o \in \mathbb{O}(G)} a_{r^*(o), o}.$$

Combining the last two results yields (17), which completes the proof. ■

As mentioned in footnote 17, the following lemma shows that restricting attention to decompositions with a finite number of buyer and seller replicas is without loss of generality.

Lemma 3 *A buyer b who can be decomposed into any number of replicas can also be decomposed into O replicas. A seller s who can be decomposed into any number of replicas can also be decomposed into O_s replicas.*

Proof of Lemma 3: Consider a buyer b and a set of objects \mathbb{O} . Let A^b be a decomposition of b containing any number of rows. $\hat{\mathbb{B}}_b$ is the set of replicas of b , each represented as a row of

A^b . Therefore, A^b has $\hat{B}_b \equiv |\hat{\mathbb{B}}_b|$ rows and O columns. We need to show that for any number \hat{B}_b , there exists an $O \times O$ matrix that constitutes a decomposition of b . The case for a seller is analogous except that the decomposition only has O_s columns. Hence we only prove the lemma for a buyer.

If $\hat{B}_b \leq O$, then it is possible to add $O - \hat{B}_b$ rows of zeroes without changing the problem, so any buyer that can be decomposed into $\hat{B}_b \leq O$ replicas can also be decomposed into O replicas.

If $\hat{B}_b > O$, then objects are on the short side and any optimal matching will leave at least $\hat{B}_b - O$ replicas unmatched. Let $\hat{\mathbb{B}}'_b \subseteq \hat{\mathbb{B}}_b$ be a subset containing $\hat{B}_b - O$ replicas that are unmatched in an optimal matching of A^b . Let $\hat{A}^b := A^b_{-\hat{\mathbb{B}}'_b}$ denote the submatrix of A^b where the rows corresponding to replicas in $\hat{\mathbb{B}}'_b$ have been removed. Because \hat{A}^b contains O rows, the proof is complete if the following can be shown:

$$V^*(A^b_{x_b}) = V^*(\hat{A}^b_{x_b}) \quad \text{for all } x_b \in \mathbb{P}.$$

By Lemma 2(v), any subset of replicas and any subset of objects are complements to each other in assignment game A^b . As $\hat{\mathbb{B}}'_b \subseteq \hat{\mathbb{B}}_b$ and $\mathbb{O} \setminus x_b \subseteq \mathbb{O}$ for any $x_b \in \mathbb{P}$, Lemma 2(v) implies

$$V^*(A^b) - V^*(\hat{A}^b) + V^*(A^b) - V^*(A^b_{x_b}) \geq V^*(A^b) - V^*(\hat{A}^b_{x_b}),$$

which is equivalent to

$$V^*(A^b) - V^*(\hat{A}^b) \geq V^*(A^b_{x_b}) - V^*(\hat{A}^b_{x_b}).$$

Because the replicas in $\hat{\mathbb{B}}'_b$ are optimally unmatched in A^b , $V^*(A^b) = V^*(\hat{A}^b)$ and the left side of the above inequality is equal to zero. Because removing rows always weakly reduces the value of an assignment game, the right side is also equal to zero. It follows that $V^*(A^b_{x_b}) = V^*(\hat{A}^b_{x_b})$ for all $x_b \in \mathbb{P}$, and the proof is complete. ■

Proof of Theorem 4: (If) Suppose that the allocation problem is such that all buyers and all sellers are decomposable. Then it is possible to construct a $(BO + O) \times O$ matrix A such

that the rows devoted to replicas of each agent are those of his decomposition. That is,

$$A := \begin{pmatrix} & A^{b_1} & & & & \\ & A^{b_2} & & & & \\ & A^{b_3} & & & & \\ & \vdots & & & & \\ & A^{b_B} & & & & \\ A^{s_1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \\ \mathbf{0} & A^{s_2} & \mathbf{0} & \cdots & \mathbf{0} & \\ \mathbf{0} & \mathbf{0} & A^{s_3} & \cdots & \mathbf{0} & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & A^{s_S} & \end{pmatrix},$$

where A^{b_i} is the decomposition of the i th buyer and A^{s_j} the decomposition of the j th seller. Observe that A satisfies Definition 1 by construction. Consider a feasible allocation x and recall that $\mathbb{L}(A, x)$ is the set of feasible matchings of A that are isomorphic to x . Notice that the elements of $\mathbb{L}(A, x)$ only differ in the way they assign objects across replicas of the same individual. A best isomorphic matching of x in A , L^x , is one that, for each buyer b assigns the objects in x_b to replicas in $\hat{\mathbb{B}}_b$ optimally and for each seller s assigns the objects in $\mathbb{O}_s \setminus x_b$ to replicas in $\hat{\mathbb{S}}_s$ optimally. (If this were not the case, then another isomorphic matching would exist that produces a larger value, contradicting that L^x is a best isomorphic matching.) This implies the first equality signs in each of the following two lines.

$$\begin{aligned} V^*(A_{x_b}^b) &= \sum_{r \in \hat{\mathbb{B}}_b} \sum_{o \in x_b} l_{r,o}^x a_{r,o} = \sum_{r \in \hat{\mathbb{B}}_b} \sum_{o \in \mathbb{O}} l_{r,o}^x a_{r,o} & \text{for all } b \in \mathbb{B} & \text{ and} \\ V^*(A_{\mathbb{O}_s \setminus x_s}^s) &= \sum_{r \in \hat{\mathbb{S}}_s} \sum_{o \in \mathbb{O}_s \setminus x_s} l_{r,o}^x a_{r,o} = \sum_{r \in \hat{\mathbb{S}}_s} \sum_{o \in \mathbb{O}} l_{r,o}^x a_{r,o} & \text{for all } s \in \mathbb{S}. \end{aligned}$$

Each of the second two equality signs derives from the fact that L^x is isomorphic to x , which means (i) $l_{r,o}^x = 0$ for all $r \in \hat{\mathbb{B}}_b$ and $o \in \mathbb{O}$ such that $o \notin x_b$ and (ii) $l_{r,o}^x = 0$ for all $r \in \hat{\mathbb{S}}_s$ and $o \in \mathbb{O}$ such that $o \notin \mathbb{O}_s \setminus x_s$.

Because this is true for any feasible allocation, it also holds for the empty allocation x^0 and its best isomorphic matching L^{x^0} :

$$V^*(A^s) = \sum_{r \in \hat{\mathbb{S}}_s} \sum_{o \in \mathbb{O}_s} l_{r,o}^{x^0} a_{r,o} = \sum_{r \in \hat{\mathbb{S}}_s} \sum_{o \in \mathbb{O}} l_{r,o}^{x^0} a_{r,o} \quad \text{for all } s \in \mathbb{S}.$$

Combining these results with the assumption that all agents are decomposable yields

$$\begin{aligned} \bar{u}_b(x_b, \mathbf{v}_b) &= V^*(A_{x_b}^b) = \sum_{r \in \hat{\mathbb{B}}_b} \sum_{o \in \mathbb{O}} l_{r,o}^x a_{r,o} & \text{for all } b \in \mathbb{B} & \text{ and} \\ \underline{v}_s(x_s, \mathbf{c}_s) &= V^*(A^s) - V^*(A_{\mathbb{O}_s \setminus x_s}^s) = \sum_{r \in \hat{\mathbb{S}}_s} \sum_{o \in \mathbb{O}} [l_{r,o}^{x^0} a_{r,o} - l_{r,o}^x a_{r,o}] & \text{for all } s \in \mathbb{S}. \end{aligned}$$

That is, Definition 2 is satisfied, and so the allocation problem is a matching problem. \square

(Only If) Suppose that a buyer b is not decomposable. Then for any matrix A^b with O replicas of b in its rows and O objects in its columns there exists a feasible allocation x such that $\bar{u}_b(x_b, \mathbf{v}_b) \neq V^*(A^b_{x_b})$. Additionally, for any $(BO + O) \times O$ replica-object assignment game A where the replicas of b are represented by the rows of A^b , a best isomorphic matching L^x of x , optimally assigns objects in x_b to replicas of b . That is,

$$\sum_{r \in \mathbb{B}_b} \sum_{o \in \mathbb{O}} l_{r,o}^x a_{r,o} = \sum_{r \in \mathbb{B}_b} \sum_{o \in x_b} l_{r,o}^x a_{r,o} = V^*(A^b_{x_b}) \neq \bar{u}_b(x_b, \mathbf{v}_b).$$

Consequently the allocation problem is not a matching problem.

Likewise, if a seller s is not decomposable, then for any $O_s \times O_s$ matrix A^s there exists a feasible allocation x such that $\underline{\kappa}_s(x_s, \mathbf{c}_s) \neq V^*(A^s) - V^*(A^s_{\mathbb{O}_s \setminus x_s})$. Additionally, for any $(BO + O) \times O$ replica-object assignment game A where the replicas of s are represented by the rows of A^s , L^x , a best isomorphic matching of x , optimally assigns objects in $\mathbb{O}_s \setminus x_s$ to replicas of s . That is,

$$\sum_{r \in \mathbb{S}_s} \sum_{o \in \mathbb{O}} l_{r,o}^{x^0} a_{r,o} - l_{r,o}^x a_{r,o} = \sum_{r \in \mathbb{S}_s} \sum_{o \in \mathbb{O}_s} l_{r,o}^{x^0} a_{r,o} - \sum_{r \in \mathbb{S}_s} \sum_{o \in \mathbb{O}_s \setminus x_s} l_{r,o}^x a_{r,o} = V^*(A^s) - V^*(A^s_{\mathbb{O}_s \setminus x_s}) \neq \underline{\kappa}_s(x_s, \mathbf{c}_s),$$

Thus the allocation problem is not a matching problem. \blacksquare

Proof of Proposition 2: (Buyers) Let b be a buyer whose payoff function exhibits RDD.

Then there exists a partition $\{\mathbb{O}_m\}_{m \in \mathbb{M}^b}$ of \mathbb{O} such that $\mathbb{O}_m = \{o_{(1)}(\mathbb{O}_m), o_{(2)}(\mathbb{O}_m), \dots, o_{(O_m)}(\mathbb{O}_m)\}$ for all $m \in \mathbb{M}^b$. For any package $x \in \mathbb{P}$, recall that $x_m \equiv x \cap \mathbb{O}_m$ and let $v_{(i)}(x_m) := \bar{u}_b(\{o_{(i)}(x_m)\}, \mathbf{v}_b)$ be the i th highest stand-alone utility among objects in x_m . For all $m \in \mathbb{M}^b$, define

$$A^m := \begin{pmatrix} v_{(1)}(\mathbb{O}_m) & v_{(2)}(\mathbb{O}_m) & \dots & v_{(O_m)}(\mathbb{O}_m) \\ \max\{v_{(1)}(\mathbb{O}_m) - \delta_{m,2}, 0\} & \max\{v_{(2)}(\mathbb{O}_m) - \delta_{m,2}, 0\} & \dots & \max\{v_{(O_m)}(\mathbb{O}_m) - \delta_{m,2}, 0\} \\ \vdots & \vdots & \ddots & \vdots \\ \max\{v_{(1)}(\mathbb{O}_m) - \delta_{m,O}\} & \max\{v_{(2)}(\mathbb{O}_m) - \delta_{m,O}, 0\} & \dots & \max\{v_{(O_m)}(\mathbb{O}_m) - \delta_{m,O}, 0\} \end{pmatrix}$$

and let

$$A := \begin{pmatrix} A^{m_1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & A^{m_2} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & A^{m_{M^b}} \end{pmatrix}.$$

We prove that b is decomposable by showing that A constitutes a decomposition of b . Because all other elements are zeroes, for any $x \in \mathbb{P}$, the value of A_x (which as may be recalled is the

submatrix of A containing exclusively its columns related to objects in x) is maximized by separately maximizing the values of all submatrices $A_{x_m}^m$. That is,

$$V^*(A_x) = \sum_{m \in \mathbb{M}^b} V^*(A_{x_m}^m).$$

In order to complete the proof, it remains to show that

$$V^*(A_{x_m}^m) = \sum_{i=1}^{|x_m|} \max\{\bar{u}_b(\{o_{(i)}(x_m)\}, \mathbf{v}_b) - \delta_{m,i}, 0\} \quad \text{for all } m \in \mathbb{M}^b \text{ and all } x \in \mathbb{P}.$$

For some $m \in \mathbb{M}^b$ and some $x \in \mathbb{P}$, consider

$$A_{x_m}^m = \begin{pmatrix} v_{(1)}(x_m) & v_{(2)}(x_m) & \dots & v_{(|x_m|)}(x_m) \\ \max\{v_{(1)}(x_m) - \delta_{m,2}, 0\} & \max\{v_{(2)}(x_m) - \delta_{m,2}, 0\} & \dots & \max\{v_{(|x_m|)}(x_m) - \delta_{m,2}, 0\} \\ \vdots & \vdots & \ddots & \vdots \\ \max\{v_{(1)}(x_m) - \delta_{m,|x|}, 0\} & \max\{v_{(2)}(x_m) - \delta_{m,|x|}, 0\} & \dots & \max\{v_{(|x_m|)}(x_m) - \delta_{m,|x|}, 0\} \\ \vdots & \vdots & \ddots & \vdots \\ \max\{v_{(1)}(x_m) - \delta_{m,O_m}, 0\} & \max\{v_{(2)}(x_m) - \delta_{m,O_m}, 0\} & \dots & \max\{v_{(|x_m|)}(x_m) - \delta_{m,O_m}, 0\} \end{pmatrix}.$$

Because the discount parameters are nondecreasing ($\delta_{m,i} \leq \delta_{m,j}$ for $i < j$), the last $O - |x_m|$ rows of $A_{x_m}^m$ are optimally unmatched. Additionally, the max function implies that, for $i = 1, \dots, |x_m|$, the i th row of $A_{x_m}^m$ be optimally matched with the i th column. In this way, the highest value is matched with the lowest discount, then the second-highest value with the second-lowest discount, and so on as long as the next value is greater than or equal to the next discount. It is easy to see that this is optimal once one notices that the problem is isomorphic to the problem of a Walrasian auctioneer facing unit demand buyers with values $v_{(1)}(x_m), \dots, v_{(1)}(|x_m|)$ and unit supply sellers with costs $\delta_{m,1}, \dots, \delta_{m,O_m}$. It follows that

$$V^*(A_{x_m}^m) = \sum_{i=1}^{|x_m|} \max\{v_{(i)}(x_m) - \delta_{m,i}, 0\} \equiv \sum_{i=1}^{|x_m|} \max\{\bar{u}_b(\{o_{(i)}(x_m)\}, \mathbf{v}_b) - \delta_{m,i}, 0\}. \quad \square$$

(Sellers) Let s be a seller whose payoff function exhibits RDD. Then there exists a partition $\{\mathbb{O}_{m,s}\}_{m \in \mathbb{M}^s}$ of \mathbb{O}_s such that $\mathbb{O}_{m,s} = \{o_{(1)}(\mathbb{O}_{m,s}), o_{(2)}(\mathbb{O}_{m,s}), \dots, o_{(O_{m,s})}(\mathbb{O}_{m,s})\}$ for all $m \in \mathbb{M}^s$. For any package $x \in \mathbb{P}_s$, recall that $x_m \equiv x \cap \mathbb{O}_{m,s}$ and let $\kappa_{(i)}(x_m) := \underline{\kappa}_s(\{o_{(i)}(x_m)\}, \mathbf{c}_s)$ be the i th lowest stand-alone cost among objects in x_m . For all $m \in \mathbb{M}^s$, define

$$A^m := \begin{pmatrix} \kappa_{(1)}(\mathbb{O}_m) & \kappa_{(2)}(\mathbb{O}_m) & \dots & \kappa_{(O_m)}(\mathbb{O}_m) \\ \kappa_{(1)}(\mathbb{O}_m) + \delta_{m,2} & \kappa_{(2)}(\mathbb{O}_m) + \delta_{m,2} & \dots & \kappa_{(O_m)}(\mathbb{O}_m) - \delta_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_{(1)}(\mathbb{O}_m) - \delta_{m,O} & \kappa_{(2)}(\mathbb{O}_m) - \delta_{m,O} & \dots & \kappa_{(O_m)}(\mathbb{O}_m) - \delta_{m,O} \end{pmatrix}$$

and let

$$A := \begin{pmatrix} A^{m_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & A^{m_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & A^{m_{M^s}} \end{pmatrix}.$$

For any $m \in \mathbb{M}^s$, consider

$$A_{\mathbb{O}_{m,s} \setminus x_m}^m = \begin{pmatrix} \kappa_{(1)}(\mathbb{O}_m \setminus x_m) & \kappa_{(2)}(\mathbb{O}_m \setminus x_m) & \cdots & \kappa_{(\mathbb{O}_m - |x_m|)}(\mathbb{O}_m \setminus x_m) \\ \kappa_{(1)}(\mathbb{O}_m \setminus x_m) + \delta_{m,2} & \kappa_{(2)}(\mathbb{O}_m \setminus x_m) + \delta_{m,2} & \cdots & \kappa_{(\mathbb{O}_m - |x_m|)}(\mathbb{O}_m \setminus x_m) - \delta_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_{(1)}(\mathbb{O}_m \setminus x_m) + \delta_{m, \mathbb{O}_{m,s}} & \kappa_{(2)}(\mathbb{O}_m \setminus x_m) + \delta_{m, \mathbb{O}_{m,s}} & \cdots & \kappa_{(\mathbb{O}_m - |x_m|)}(\mathbb{O}_m \setminus x_m) + \delta_{m, \mathbb{O}_{m,s}} \end{pmatrix},$$

the matrix formed by removing all columns related to an object in x_m from A^m . Because the discount parameters are nondecreasing ($\delta_{m,i} \leq \delta_{m,j}$ for $i < j$), the first $|x_m|$ rows of $A_{\mathbb{O}_{m,s} \setminus x_m}^m$ optimally remain unmatched. Any assignment that matches each of the remaining $\mathbb{O}_{m,s} - |x_m|$ rows to one of the $\mathbb{O}_{m,s} - |x_m|$ columns produces the same value and is therefore optimal. It follows that

$$V^*(A_{\mathbb{O}_{m,s} \setminus x_m}^m) = \sum_{i=1}^{\mathbb{O}_m - |x_m|} \kappa_{(i)}(\mathbb{O}_m \setminus x_m) + \sum_{i=|x_m|+1}^{\mathbb{O}_m} \delta_{m,i} \quad \text{for all } m \in \mathbb{M}^s \text{ and all } x \in \mathbb{P}_s.$$

Because all other elements are zeroes, for any $x \in \mathbb{P}_s$, the value of $A_{\mathbb{O}_s \setminus x}$ is maximized by separately maximizing the values of all submatrices $A_{\mathbb{O}_{m,s} \setminus x_m}^m$. That is,

$$V^*(A_{\mathbb{O}_s \setminus x}) = \sum_{m \in \mathbb{M}^s} V^*(A_{\mathbb{O}_{m,s} \setminus x_m}^m) = \sum_{m \in \mathbb{M}^s} \left[\sum_{i=1}^{\mathbb{O}_m - |x_m|} \kappa_{(i)}(\mathbb{O}_m \setminus x_m) + \sum_{i=|x_m|+1}^{\mathbb{O}_m} \delta_{m,i} \right] \quad \text{for all } x \in \mathbb{P}_s.$$

It follows that, for all $x \in \mathbb{P}_s$,

$$\begin{aligned} V^*(A) - V^*(A_{\mathbb{O}_s \setminus x}) &= \sum_{m \in \mathbb{M}^s} \left[\sum_{i=1}^{\mathbb{O}_m} \kappa_{(i)}(\mathbb{O}_m) + \sum_{i=1}^{\mathbb{O}_m} \delta_{m,i} - \sum_{i=1}^{\mathbb{O}_m - |x_m|} \kappa_{(i)}(\mathbb{O}_m \setminus x_m) - \sum_{i=|x_m|+1}^{\mathbb{O}_m} \delta_{m,i} \right] \\ &= \sum_{m \in \mathbb{M}^s} \sum_{i=1}^{|x_m|} [\kappa_{(i)}(x_m) + \delta_{m,i}] \\ &\equiv \sum_{m \in \mathbb{M}^s} \sum_{i=1}^{|x_m|} [\underline{\kappa}_s(o_{(i)}(x_m), \mathbf{c}_s) + \delta_{m,i}] \\ &= \underline{\kappa}_s(x, \mathbf{c}_s). \quad \blacksquare \end{aligned}$$

Proof of Proposition 3: (Buyers) Consider a buyer b consuming package $x \in \mathbb{P}$ whose utility function is such that y and z with $y, z \subseteq x$ and $y \cap z = \emptyset$ are strict complements to each

other. That is,

$$\bar{u}_b(x, \mathbf{v}_b) - \bar{u}_b(x \setminus y, \mathbf{v}_b) > \bar{u}_b(x \setminus z, \mathbf{v}_b) - \bar{u}_b(x \setminus (y \cup z), \mathbf{v}_b).$$

Suppose b is decomposable and let A be a decomposition. Writing $\tilde{A} := A_x$ (the matrix containing only those columns of A that correspond to an object in x) for ease of notation, the above inequality is equivalent to

$$V^*(\tilde{A}) - V^*(\tilde{A}_{\cdot, -y}) > V^*(\tilde{A}_{\cdot, -z}) - V^*(\tilde{A}_{\cdot, -(y \cup z)}),$$

The latter inequality states that subsets of objects y and z are complements to each other in assignment game \tilde{A} , which contradicts part (iii) of Lemma 2. \square

(Sellers) Consider a seller s producing package $x \in \mathbb{P}_s$ and whose cost function is such that $y, z \subseteq x$ with $y \cap z = \emptyset$ are strict complements to each other. Then

$$\underline{\kappa}_s(x, \mathbf{c}_s) - \underline{\kappa}_s(x \setminus y_s, \mathbf{c}_s) < \underline{\kappa}_s(x \setminus z_s, \mathbf{c}_s) - \underline{\kappa}_s(x \setminus (y \cup z), \mathbf{c}_s).$$

Suppose s is decomposable and let A be a decomposition. Writing $\tilde{x} := x \setminus (y \cup z)$ and $\tilde{A} := A_{\mathbb{O}_s \setminus \tilde{x}}$, the above inequality is equivalent to

$$\begin{aligned} (V^*(A) - V^*(\tilde{A}_{\cdot, -(y \cup z)})) - (V^*(A) - V^*(\tilde{A}_{\cdot, -z})) &< (V^*(A) - V^*(\tilde{A}_{\cdot, -y})) - (V^*(A) - V^*(\tilde{A})) \\ \Leftrightarrow V^*(\tilde{A}_{\cdot, -z}) - V^*(\tilde{A}_{\cdot, -(y \cup z)}) &> V^*(\tilde{A}) - V^*(\tilde{A}_{\cdot, -y}). \end{aligned}$$

The latter inequality states that subsets of objects y and z are complements to each other in assignment game \tilde{A} , which contradicts part (iii) of Lemma 2. \blacksquare

B Extension to Bayesian Mechanisms

For the purposes of this section, we assume that the types \mathbf{v}_b and \mathbf{c}_s are independently distributed across agents according to a probability measure with full support on V_b and C_s , respectively.

We introduce two additional conditions that allow us to extend Theorem 1 to Bayesian mechanisms. The mechanism $(\chi, \mathbf{t}^\beta, \mathbf{t}^\sigma)$ is **Bayesian incentive compatible (BIC)** if for all b and v_b ,

$$E_{\mathbf{v}_{-b}, \mathbf{c}} \left[u_b(\chi(\mathbf{v}, \mathbf{c}), \mathbf{v}_b) - t_b^\beta(\mathbf{v}, \mathbf{c}) \right] = \sup_{\hat{\mathbf{v}}_b} \left\{ E_{\mathbf{v}_{-b}, \mathbf{c}} \left[u_b(\chi(\hat{\mathbf{v}}_b, \mathbf{v}_{-b}, \mathbf{c}), \mathbf{v}_b) - t_b^\beta(\hat{\mathbf{v}}_b, \mathbf{v}_{-b}, \mathbf{c}) \right] \right\},$$

and for all s and c_s ,

$$E_{\mathbf{v}, \mathbf{c}_{-s}} [t_s^\sigma(\mathbf{v}, \mathbf{c}) - \kappa_s(\chi(\mathbf{v}, \mathbf{c}), \mathbf{c}_s)] = \sup_{\hat{\mathbf{c}}_s} \{E_{\mathbf{v}, \mathbf{c}_{-s}} [t_s^\sigma(\mathbf{v}, \hat{\mathbf{c}}_s, \mathbf{c}_{-s}) - \kappa_s(\chi(\mathbf{v}, \hat{\mathbf{c}}_s, \mathbf{c}_{-s}), \mathbf{c}_s)]\}.$$

The mechanism is **(interim) individually rational (IR)** if for all b and v_b ,

$$E_{\mathbf{v}_{-b}, \mathbf{c}} \left[u_b(\chi(\mathbf{v}, \mathbf{c}), \mathbf{v}_b) - t_b^\beta(\mathbf{v}, \mathbf{c}) \right] \geq 0,$$

and for all s and c_s ,

$$E_{\mathbf{v}, \mathbf{c}_{-s}} [t_s^\sigma(\mathbf{v}, \mathbf{c}) - \kappa_s(\chi(\mathbf{v}, \mathbf{c}), \mathbf{c}_s)] \geq 0.$$

For the purposes of Theorem 5, we impose the additional assumption that **revenue equivalence** holds.²⁰ That is, we assume that if the mechanism $(\chi, \mathbf{t}^\beta, \mathbf{t}^\sigma)$ is BIC, then the expected payoff for each player is determined by χ up to an additive constant. As shown in Krishna and Maenner (2001), this condition holds if for each b , the set of types V_b is convex and $u_b(x_b, \cdot)$ is a convex function, and for each s , the set of types C_s is convex and $-\kappa_s(x_s, \cdot)$ is a convex function. As they argue, this condition is satisfied in all standard auction design problems (Krishna and Maenner, 2001, p.1115).

Theorem 5 *If buyers and sellers are component complements and revenue equivalence holds, then there is no ex post efficient BIC-IR mechanism that runs a budget surplus in expectation. Buyers and sellers are strict component complements, then every ex post efficient BIC-IR mechanism runs a budget deficit in expectation.*

Proof: Let $m = (\chi, \mathbf{t}^\beta, \mathbf{t}^\sigma)$ be a mechanism satisfying incentive compatibility and individual rationality such that the allocation rule χ is ex post efficient. Let m^* be the VCG mechanism based on allocation rule χ . The incentive compatibility of m and m^* and the assumption of revenue equivalence imply that m has the same payment rule as m^* up to a constant. Consider a buyer of type $\underline{\mathbf{v}}_b$ and a seller s of type $\bar{\mathbf{c}}_s$. By the definition of a VCG mechanism, for mechanism m^* , payments from and to such types of agents are 0, and so are their payoffs because $W_{-b, \cdot}^* = W^*$ and $W_{\cdot, -s}^* = W^*$ for any b and s that do not trade. By the individual rationality of m , the expected payments from and to these agents under mechanism m must

²⁰A first instance of revenue equivalence was noticed by Vickrey (1961, 1962). Myerson (1981) and Riley and Samuelson (1981) provide general formulations and formalization. The revenue equivalence theorem we are invoking here is due to Krishna and Maenner (2001); see also Krishna (2002). Proofs of impossibility results using revenue equivalence were developed by Williams (1999) and independently by Krishna and Perry (2000), based on arguments first developed Makowski and Mezzetti (1994). For an alternative approach and generalization, see Makowski and Ostroy (1989) and the extension by Segal and Whinston (2012).

be nonnegative. Thus, \mathbf{t}^β is equal to the VCG payment rule plus a nonnegative constant and \mathbf{t}^σ is equal to the VCG payment rule plus a nonnegative constant. It follows that m generates weakly less revenue than $R(\mathbf{v}, \mathbf{c})$ when the types are (\mathbf{v}, \mathbf{c}) . Thus, to prove the theorem, it is sufficient to show that $R(\mathbf{v}, \mathbf{c}) \leq 0$ for all (\mathbf{v}, \mathbf{c}) and that $R(\mathbf{v}, \mathbf{c}) < 0$ whenever (CC) holds strictly, which follows from Theorem 1. ■

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